ATLANTA PUBLIC SCHOOLS
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Atlanta Public Schools

Teacher's Curriculum Supplement

Mathematics II: Unit 5
Piecewise, Inverse, and Exponential Functions

GE Foundation

This document has been made possible by funding from the GE Foundation Developing Futures grant, in partnership with Atlanta Public Schools. It is derived from the Georgia Department of Education Math II Framework and includes contributions from Georgia teachers. It is intended to serve as a companion to the GA DOE Math II Framework Teacher Edition. Permission to copy for educational purposes is granted and no portion may be reproduced for sale or profit.

Preface

We are pleased to provide this supplement to the Georgia Department of Education's Mathematics II Framework. It has been written in the hope that it will assist teachers in the planning and delivery of the new curriculum, particularly in these first years of implementation. This document should be used with the following considerations.

- The importance of working the tasks used in these lessons cannot be overstated. In planning for the teaching of the Georgia Performance Standards in Mathematics teachers should work the tasks, read the teacher notes provided in the Georgia Department of Education's Mathematics II Framework Teacher Edition, and *then* examine the lessons provided here.

- This guide provides day-by-day lesson plans. While a detailed scope and sequence and established lessons may help in the implementation of a new and more rigorous curriculum, it is hoped that teachers will assess their students informally on an on-going basis and use the results of these assessments to determine (or modify) what happens in the classroom from one day to the next. Planning based on student need is much more effective than following a pre-determined timeline.

- It is important to remember that the Georgia Performance Standards provide a balance of concepts, skills, and problem solving. Although this document is primarily based on the tasks of the Framework, we have attempted to help teachers achieve this all important balance by embedding necessary skills in the lessons and including skills in specific or suggested homework assignments. The teachers and writers who developed these lessons, however, are not in your classrooms. It is incumbent upon the classroom teacher to assess the skill level of students on every topic addressed in the standards and provide the opportunities needed to master those skills.

- In most of the lesson templates, the sections labeled *Differentiated support/enrichment* have been left blank. This is a result of several factors, the most significant of which was time. It is hoped that as teachers use these lessons, they will contribute their own ideas, not only in the areas of differentiation and enrichment, but in other areas as well. Materials and resources abound that can be used to contribute to the teaching of the standards.

On the topic of differentiation, it is critical to reiterate that many of the strategies used in a standards-based mathematics classroom promote differentiation. These strategies include

- the use of rich tasks with multiple points of entry and more than one path to a solution,
- flexible grouping of students,
- multiple representations of mathematical concepts,
- writing in mathematics,
- monitoring of progress through on-going informal and formative assessments, and
- analysis of student work.

We hope that teachers will incorporate these strategies in each and every lesson.

It is hoped that you find this document useful as we strive to raise the mathematics achievement of all students in our district and state. Comments, questions, and suggestions for inclusions related to this document may be emailed to Dr. Dottie Whitlow, Executive Director, Mathematics and Science Department, Atlanta Public Schools, dwhitlow@atlantapublicschools.us.

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Explanation of the Terms and Categories Used in the Lesson Template

Task: This section gives the suggested number of days needed to teach the concepts addressed in a task, the task name, and the problem numbers of the task as listed in the Georgia Department of Education's Mathematics II Framework Teacher Edition (GaDOE TE).

In some cases new tasks or activities have been developed. These activities have been named by the writers.

Standard(s): Although each task addresses many Math II standards and uses mathematics learned in earlier grades, in this section, only the key standards addressed in the lesson are listed.

New Vocabulary: Vocabulary is listed here the *first* time it is used. It is strongly recommended that teachers, particularly those teaching Math Support, use interactive word walls. Vocabulary listed in this section should be included on the word walls and previewed in Math Support.

Mathematical concepts/skills: Major concepts addressed in the lesson are listed in this section whether they are Math II concepts or were addressed in earlier grades or courses.

Prior knowledge: Prior knowledge includes only those topics studied in previous grades or courses. It does not include Math II content taught in previous lessons.

Essential Question(s): Essential questions may be daily and/or unit questions.

Suggested materials: This is an attempt to list all materials that will be needed for the lesson, including consumable items, such as graph paper; and tools, such as graphing calculators and compasses. This list does not include those items that should always be present in a standards-based mathematics classroom such as markers, chart paper, and rulers.

Warm-up: A suggested warm-up is included with every lesson. Warm-ups should be brief and should focus student thinking on the concepts that are to be addressed in the lesson.

Opening: Openings should set the stage for the mathematics to be done during the work time. The amount of class time used for an opening will vary from day-to-day but this should not be the longest part of the lesson.

Worktime: The problem numbers have been listed and the work that students are to do during the worktime has been described. A student version of the day's activity follows the lesson template in every case. In order to address all of the standards in Math II, some of the problems in some of the original GaDOE tasks have been omitted and less time consuming activities have been substituted for those problems. In many instances, in the student versions of the tasks, the writing of the original tasks has been simplified. In order to preserve all vocabulary, content, and meaning it is important that teachers work the original tasks as well as the student versions included here.

Teachers are expected to both facilitate and provide some direct instruction, when necessary, during the work time. Suggestions related to student misconceptions, difficult concepts, and deeper meaning have been included in this section. However, the teacher notes in the GaDOE Math II Framework are exceptional. In most cases, there is no need to repeat the information provided there. Again, it is imperative that teachers work the tasks and read the teacher notes that are provided in GaDOE support materials.

Questioning is extremely important in every part of a standards-based lesson. We included suggestions for questions in some cases but did not focus on providing good questions as extensively as we would have liked. Developing good questions related to a specific lesson should be a focus of collaborative planning time.

Closing: The closing may be the most important part of the lesson. This is where the mathematics is formalized. Even when a lesson must be continued to the next day, teachers should stop, leaving enough time to "close", summarizing and formalizing what students have done to that point. As much as possible students should assist in presenting the mathematics learned in the lesson. The teacher notes are all important in determining what mathematics should be included in the closing.

Homework: In some cases, homework suggestions are provided. Teachers should use their resources, including the textbook, to assign homework that addresses the needs of their students.

Homework should be focused on the skills and concepts presented in class, relatively short (30 to 45 minutes), and include a balance of skills and thought-provoking problems.

Differentiated support/enrichment: On the topic of differentiation, it is critical to reiterate that many of the strategies used in a standards-based mathematics classroom promote differentiation. These strategies include

- the use of rich tasks with multiple points of entry and more than one path to a solution,
- flexible grouping of students,
- multiple representations of mathematical concepts,
- writing in mathematics,
- monitoring of progress through on-going informal and formative assessments, and
- analysis of student work.

Check for understanding: A check for understanding is a short, focused assessment-a ticket out the door, for example. There are many good resources for these items, including the GaDOE culminating task at the end of each unit and the *Mathematics II End-of-Course Study Guide*. Both resources can be found on-line at www.georgiastandards.org, along with other GaDOE materials related to the standards. Problem numbers from the GaDOE culminating task have been listed with the appropriate lessons in this document.

Resources/materials for Math Support: Again, in some cases, we have provided materials and/or suggestions for Math Support. This section should be personalized to your students, class, and/or school, based on your resources.

Table of Contents

Mathematics II
Unit 5

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Unit 1 Timeline

Task 1: Planning a Race Strategy	3 days
Task 2: Parking Deck Pandemonium	3 days
Task 3: Functions and Their Inverses	3 days
Task 4: Who Wants to Be a Millionaire?	2 days
Task 5: Beginning a Business	1 day

Task Notes

The importance of working the tasks used in these lessons cannot be overstated. In planning for the teaching of the Georgia Performance Standards in Mathematics, teachers should work the Student Tasks, read any corresponding teacher notes provided in the Georgia Department of Education's Mathematics II Framework Teacher Edition, and *then* examine the lessons provided here.

The tasks provided in this Supplement are based on the content of Unit 5 of the Georgia Department of Education's Mathematics II Framework. We suggest, as always, that teachers use this Supplement along with the GaDOE Teacher Edition and the *Mathematics II End-of-Course Study Guide* which can be found on-line at www.georgiastandards.org.

Task 1: Planning a Race Strategy

The big ideas presented in this task include

- writing piecewise functions to model a given situation;
- graphing piecewise functions;
- writing piecewise functions using function notation;
- interpreting characteristics of the graphs of piecewise functions in the context of a given situation;
- investigating domains, ranges, vertices, intercepts, axes of symmetry, constant intervals, intervals of increase and decrease, extrema, and rates of change of piecewise functions;
- writing absolute value functions as piecewise functions; and
- using appropriate technology to graph piecewise functions.

All parts of the original GaDOE task have been used in these lessons. Some items have been renumbered.
Original DOE *Item 5* has been designated as *Day 2 Homework*. Original DOE *Items 6, 9c,* and *9d* have been assigned as *Day 3 Homework*. These items are crucial to student understanding and should be thoroughly discussed in class after they have been completed.

Items 5d and 5*e* of the *Day 2 Student Task* were added to the original DOE task. These items address standard MM2A1c. Solutions are included in the *Day 2 Lesson Plan.*

Task 2: Parking Deck Pandemonium
The big ideas presented in this task include

- using step functions to model real situations;
- evaluating and graphing step functions;
- investigating characteristics of step functions, including the greatest integer function and the least integer function; and
- describing and graphing transformations of step functions.

All parts of the original GaDOE task have been used in these lessons. *Item 4* of the *Day 1 Student Task* and the short discussion on discontinuity that precedes the item were added to the original DOE task for purposes of this Supplement. It is important that students have a simple intuitive understanding of continuity and points of discontinuity at this juncture. *Item 10* of the *Day 3 Student Task* was also added. This item may be presented to the entire class or used as enrichment.

Task 3: Functions and Their Inverses
The big ideas presented in this task include

- finding the composition of two functions;
- understanding the relationship between a function and its inverse;
- finding the inverse of a given function;
- verifying that two functions are inverses;
- determining the relationship between the graphs of inverse functions; and
- determining which functions have inverses.

The original GaDOE task containing many of these items is entitled *Please Tell Me in Dollars and Cents*. Items 1, 2, 4 – 7, 9, 13 and 14 of the original task are included in this Supplement.

Items have been added to the original DOE task in order to address the topic of one-to-one functions.

Task 4: Who Wants to Be a Millionaire?
The big ideas presented in this task include

- extending properties of exponents to include all integer exponents;
- investigating characteristics of exponential functions, including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, rates of change, and end behavior;
- using basic exponential functions as models of real phenomena;
- graphing functions as transformations of $f(x) = a^x$;
- solving simple exponential equations and inequalities analytically, graphically, and by using appropriate technology;
- recognizing geometric sequences as exponential functions with domains that are whole numbers; and
- interpreting the constant ratio in a geometric sequence as the base of the associated exponential function.

This task does not appear in the GaDOE Mathematics II Framework.

Task 5: Beginning a Business
The big ideas presented in this task include

- using the standard compound interest formula;
- writing exponential functions to model real-world situations; and
- solving simple exponential equations and inequalities using technology.

This task contains items from *Parts 3* and *4* of the GaDOE task, *Growing by Leaps and Bounds*. Items have been added to the original task to address solving simple exponential equations and inequalities.

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Mathematics II: Unit 5

Task 1: Planning a Race Strategy

Mathematics II
Task 1: ***Planning a Race Strategy*** **Day 1/3**
(GaDOE TE Problems 1 and 2)

Standard(s): MM2A1. Students will investigate step and piecewise functions, including greatest integer and absolute value functions.

b. Investigate and explain characteristics of a variety of piecewise functions including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, points of discontinuity, intervals over which the function is constant, intervals of increase and decrease, and rates of change.

New vocabulary: piecewise function

Mathematical concepts/skills:

- calculating rates of change
- writing linear functions to model given situations
- graphing piecewise functions
- relating characteristics of a piecewise function (domain, range, vertices, and rates of change) to a given context

Prior knowledge:

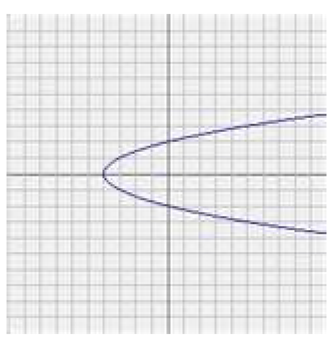
- calculating rates of change
- writing linear functions to model given situations
- using the vertical line test to determine whether a graph represents a function
- relating characteristics of a function to a given context

Essential question(s): What are piecewise functions and how can they be used to model real-world situations?

Suggested materials: graph paper

Warm-up: Post the following.

Does the graph shown here represent a function? Why or why not?



Opening: Discuss the warm-up. Students may need to revisit the definition of a function and use of the vertical line test.

Have a student read the scenario. Discuss the scenario and *Problem 1a* with students. Original GaDOE questions requiring students to convert from kilometers per minute to kilometers per hour and miles per hour have been omitted in the student task that follows. We suggest that teachers discuss these conversions during the opening in order for students to have a better understanding of the average speeds at which the runners are traveling.

Worktime: Students should work in pairs to complete *Problems 1* and *2* of the task.

Problem 1, Part d may be difficult for students. As you monitor student work, ask guiding questions that encourage different approaches. Students may take a trial and error approach using the slope and manipulating the formula to obtain the desired points; or, realizing that the function is linear, they may use a form of an equation of a line to write the function.

Have a whole class discussion of *Problem 1* before students begin *Problem 2*. Discuss different approaches used to answer *Problem 1, Part d.* Be sure to review different methods of writing an equation of a line, including slope-intercept and point-slope forms.

Note that *Problem 2* of the original GaDOE task has been revised. GaDOE Teacher Notes still apply.

Closing: Allow students to share their descriptions of how Jim should run the 5K race based on the graph given in *Problem 2*.

Homework:

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: In addition to terms listed as ***new vocabulary,*** students should preview:

- calculating rates of change
- writing linear functions to model given situations
- using the vertical line test to determine whether a graph represents a function
- relating characteristics of a function to a given context

Mathematics II
Planning a Race Strategy
Day 1 Student Task

Saundra is a personal trainer at a local gym. Three of her clients asked her to help them train for an upcoming 5K race. Based on her knowledge of the physical condition of each client, Saundra developed an individualized strategy for each to use in running the race and then designed training plans to support the strategies.

1. One of the clients is Terrance, a very experienced runner. His plan is to run at a moderate pace for the first two kilometers and then use his maximum speed for the final three kilometers.
   a. Based on data from Terrance's workouts at the gym, Saundra has determined that covering 2 kilometers in 10 minutes corresponds to a moderate pace for Terrance. What is this pace in kilometers per minute (km/min)?

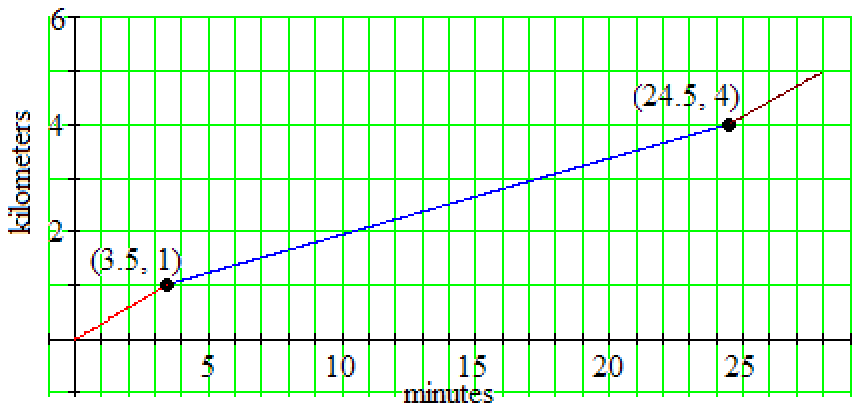
   b. Let f denote the function that expresses the distance Terrance covers after t minutes of running for times $0 \le t \le 10$. Write the formula for $f(t)$, and graph the function f by hand.

   c. Saundra believes that, with training, Terrance can run fast enough to cover the last 3 kilometers of the race in 10 minutes. What average speed is this in kilometers per minute?

   d. Let g denote the function that expresses the total distance Terrance covers after t minutes, for times $10 \le t \le 20$, assuming that the distance covered is already 2 kilometers at time $t = 10$ and that Terrance's running speed for the next 10 minutes is the speed discussed in *Part C*. Write the formula for $g(t)$, and graph the function g.

   e. The functions f and g together represent the proposed strategy for Terrance to use in running the race. Draw the graphs of f and the g on the same coordinate axes. Consider this as one graph. What is the domain of the graph? Is it the graph of a function? Why or why not?

2. Jim, the second client, is not as strong a runner as Terrance but likes to start fast. Saundra's strategy for Jim to use in running the race is shown in the graph below.



   a. Give the domain and range of the function graphed above. What does the domain tell you about Jim's race? What does the range tell you about the race?

   b. Use the graph to write a verbal description of how Jim should run the 5K race. Your description should note the points in the race at which he is supposed to change his running pace and the proposed average speed during each segment.

The graphs in *Items 1 and 2* that show Terrance's and Jim's race strategies are examples of functions that cannot be specified using a single rule of correspondence between inputs and outputs. These functions are constructed by combining two or more "pieces," each of which is a function with a restricted domain. The formal mathematical term for such a function is **piecewise function**. The pieces in the situations so far have come from the family of linear functions, but, as you will see in remaining parts of this task, the pieces can be selected from other function families as well.

Mathematics II
Task 1: ***Planning a Race Strategy*** **Day 2/3**
(GaDOE TE Problems 3 - 5)

Standard(s): MM2A1. Students will investigate step and piecewise functions, including greatest integer and absolute value functions.

b. Investigate and explain characteristics of a variety of piecewise functions including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, points of discontinuity, intervals over which the function is constant, intervals of increase and decrease, and rates of change.

MM2A3. Students will analyze quadratic functions in the forms $f(x) = ax^2 + bx + c$ and $f(x) = a(x - h)^2 + k$.

b. Graph quadratic functions as transformations of the function $f(x) = x^2$.
c. Investigate and explain characteristics of quadratic functions, including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, intervals of increase and decrease, and rates of change.

MM2A4. Students will solve quadratic equations and inequalities in one variable.

a. Solve equations graphically using appropriate technology.
b. Find real and complex solutions of equations by factoring, taking square roots, and applying the quadratic formula.

New vocabulary:

Mathematical concepts/skills:

- calculating average rates of change
- graphing piecewise functions containing linear and quadratic expressions
- using graphing technology to graph piecewise functions and functions with restricted domains
- relating characteristics of a piecewise function (domain, range, vertices, rates of change, intervals over which a function is constant, intervals of increase and decrease, maximum and minimum values, and zeros) to a given context
- solving simple linear and quadratic equations
- writing piecewise functions using function notation

Prior knowledge:

- calculating average rates of change
- solving simple linear and quadratic equations
- relating characteristics of a function to a given context

Essential question(s): What are piecewise functions and how can they be used to model real-world situations? How can I graph a piecewise function using a graphing utility?

Suggested materials: graphing calculators, graph paper

Warm-up: Post the following:

Enter the following function in your graphing calculator using the given graphing window:

$Y_1 = (2x - 2)/(\ (x \geq 2)(x \leq 8)\ )$

Xmin = -1	*Y min = -1*
Xmax = 10	*Ymax = 20*
Xscl = 1	*Yscl = 1*

(Note: You can find the inequality symbols used in the expression by entering ***2ND*** *and then* ***MATH*** *on your calculator).*

Graph the function. Copy the graph onto graph paper and explain what you see. Why do you think the calculator gives this result?

Opening: The function given in the warm-up should yield the graph of the line $y = 2x - 2$ on the restricted domain $2 \leq x \leq 8$. Allow students to share their graphs and explain why they think the calculator yields only this *piece* of the function.

After students have had an opportunity to share their thoughts, make sure they understand that the function is written in the form $y_1 = \dfrac{2x-2}{(x \geq 2)(x \leq 8)}$. The calculator returns a value of 1 for true statements of inequality and a value of 0 for inequality statements that are false. In this case, if x is less than 2, a zero is returned and 0 times any number is 0. Since the denominator is 0, the expression for the function is undefined and no graph appears in that region. The same is true when x is greater than 8. If x is between 2 and 8, inclusive, a 1 is returned for both factors in the denominator and the denominator has a value of 1. Therefore $y = 2x - 2$ is graphed over this region.

Worktime: Students should work in pairs to complete *Problem 3* and as much of P*roblem 4* as time allows.

(Note: *Problems 4* and *5* of the student task in this Supplement are suggested as homework. Students should complete *Problem 4* as *Day 2 Homework* and *Problem 6* as part of the homework to follow *Day 3*.)

Have a whole class discussion of *Problem 3* before students begin *Problem 4*.

(Note: *Problem 4* of the original GaDOE task has been renumbered as *Problem 3j*.)

Closing: Discuss *Problems 3i* and *3j* thoroughly by allowing students to share their work. (See GaDOE Teacher Notes.)

Homework: *Problem 4* (*Problem 5* of the original GaDOE task) of the Student Task. In assigning the problem, make sure students understand that distance in this case is how far Jim is from his home. *It is extremely important that this problem is discussed in class after it has been completed.*

Differentiated support/enrichment:

Check for Understanding:

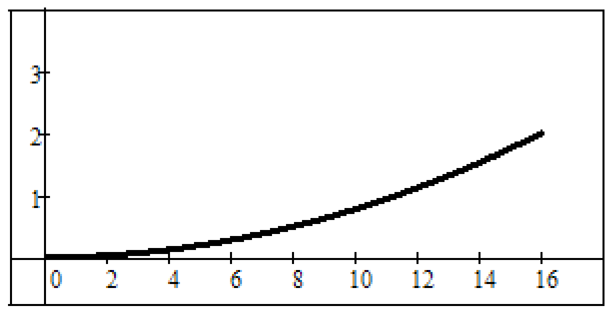
Resources/materials for Math Support: In addition to terms listed as ***new vocabulary,*** students should preview:

- calculating average rates of change of quadratic functions
- graphing quadratic functions
- using graphing technology to graph functions
- solving simple linear and quadratic equations

Mathematics II
Planning a Race Strategy
Day 2 Student Task

3. Sue is the least experienced runner among Saundra's three clients. After working with Sue for a few weeks, Saundra observed that Sue got her best time for long distances if she began slowly and steadily increased her pace until she reached a speed that she could hold for a while.

a. Saundra decided to use the function f_1 with the formula $f_1(t) = \frac{1}{128}t^2$ for $0 \le t \le 16$ to model the distance Sue might cover during the first 16 minutes of the race. Saundra entered the function
$Y_1 = (1/128)x^2 / ((0 \le x)(x \le 16))$
in her calculator and obtained the graph shown. If your graphing utility provides for logical inequality statements, use it to obtain a similar graph with the same viewing window. If it does not, state the graphing window that is needed.



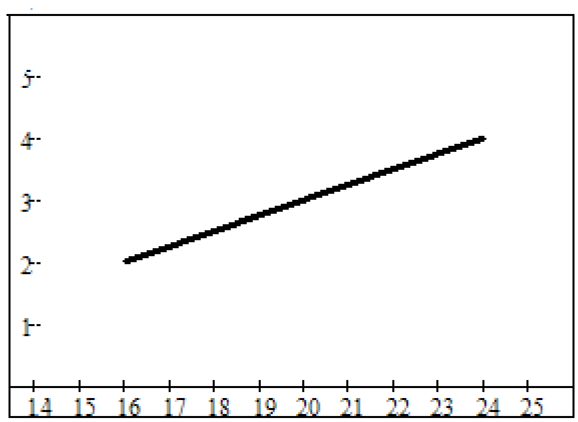
b. Explain why the function that Sue entered into her calculator graphs the function $f_1(t) = \frac{1}{128}t^2$ just for the restricted domain $0 \le t \le 16$.

c. According to the model, what distance would Sue cover in the first 16 minutes? Explain how you know.

d. Saundra projected that Sue would be running at one-fourth of a kilometer per minute by the time she had been running for 16 minutes. Saundra thought it reasonable to assume that Sue would be able to continue running at this pace for the next two kilometers. If so, how long would it take Sue to run two kilometers? How many minutes into the race would Sue be after covering these next two kilometers?

e. Saundra entered the following expression into her calculator and obtained the graph shown.
$Y_2 = ((1/4)x - 2) / ((16 < x)(x \le 24))$

If possible, use your graphing utility to obtain a similar graph with the same viewing window. If this is not possible with your graphing utility, state the viewing window.



Explain how the expression leads to the graph shown.

f. Write the function, graphed in part e, using function notation. Use f_2 for the name of the function and *t* for the input variable, and be sure to state the restriction on the domain. This function represents how Sue might run the second segment of the race.

g. Saundra decided to model the last segment of how Sue might run the race with the function f_3 defined by the formula $f_3(t) = \frac{-1}{64}(t-32)^2 + 5$. Using this model, when would Sue finish the race?

h. What is the domain for the function f_3 if this function represents the last segment of the race? Explain. Make a hand-drawn sketch of the graph of the function f_3 over this domain, and then draw the graph using your graphing utility.

i. Draw the graphs of the functions f_1, f_2, and f_3 on the same coordinate axes to form the graph of a single piecewise function that represents Saundra's model of how Sue might run the race. What feature of the graph demonstrates when Sue would speed up, slow down, or maintain a steady pace? Explain.

When we combine two or more functions to form a **piecewise function**, we use notation that indicates we have a single function. For a piecewise function *f*, we use the format:

$$f(x) = \begin{cases} \text{rule for function 1, domain for function 1} \\ \text{rule for function 2, domain for function 2} \\ \ldots \\ \text{rule for last function, domain for last function} \end{cases}.$$

j. Write the piecewise function, $Y_1 = (1/128)x^2 / ((0 \le x)(x \le 16))$, f, that corresponds to Saundra's model of how Sue might run the 5K race.

As part of his training plan, Jim runs in his neighborhood most days of the week. *Item 4* below and Day 3's *Items 5 and 6* explore functions that model three different training runs for Jim. The inputs for each of these functions represent time, as in the functions related to race strategies for Terrance, Jim, and Sue. The outputs measure distances, but there is a difference from the previous items. *Here the distance is not necessarily the distance Jim has covered since he began running; instead the distance is how far Jim is from his home.*

4. The piecewise function *d,* defined below, represents one of Jim's Saturday training runs and models his distance from home, *d*(*t*), in miles as a function of time, *t*, in minutes since he left home.

$$d(t) = \begin{cases} \frac{1}{12}t, & 0 \le t \le 30 \\ 2.5, & 30 < t < 35 \\ -\frac{1}{10}t + 6, & 35 \le t \le 60 \end{cases}$$

a. What is the domain of the function? What does the answer tell you about this particular training run?

b. Draw a graph of the function *d* on graph paper and determine the range of the function.

c. What is the maximum value of the function? Interpret your answers in relation to Jim's training run.

d. What are the zeros of the function? Interpret these in relation to Jim's training run.

e. Over what interval(s) is the function constant? Interpret your answer in relation to Jim's training run.

f. What are the intervals of increase and decrease for the function? Interpret your answers in relation to Jim's training run.

Mathematics II
Task 1: ***Planning a Race Strategy*** **Day 3/3**
(GaDOE TE Problems 7 – 9 and 6)

Standard(s): MM2A1. Students will investigate step and piecewise functions, including greatest integer and absolute value functions.

a. Write absolute value functions as piecewise functions.
b. Investigate and explain characteristics of a variety of piecewise functions including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, points of discontinuity, intervals over which the function is constant, intervals of increase and decrease, and rates of change.
c. Solve absolute value equations and inequalities analytically, graphically, and by using appropriate technology.

MM2A3. Students will analyze quadratic functions in the forms $f(x) = ax^2 + bx + c$ and $f(x) = a(x - h)^2 + k$.

a. Convert between standard and vertex form.
b. Graph quadratic functions as transformations of the function $f(x) = x^2$.
c. Investigate and explain characteristics of quadratic functions, including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, intervals of increase and decrease, and rates of change.

New vocabulary:

Mathematical concepts/skills:

- graphing transformations of the function $y = |x|$
- writing a piecewise rule for $y = |x|$
- writing piecewise rules for transformations of functions in the form $y = a|x|$
- relating characteristics of a piecewise function (domain, range, vertices, rates of change, intervals over which a function is constant, intervals of increase and decrease, maximum and minimum values, and zeros) to a given context

Prior knowledge:

- graphing transformations of the function $y = |x|$
- relating characteristics of a function to a given context

Essential question(s): How can I write absolute value functions as piecewise functions?

Suggested materials: graph paper, graphing calculator

Warm-up: Have students compare homework (*Problem 4* of the Student Task) with a partner. Ask them to be prepared to share their work with the class.

Opening: Allow students to share their responses to the homework.

Ask students to read the scenario for *Problem 5* of today's task, making notes in the margins of all pertinent information. Discuss the scenario making sure that all students understand the situation.

Worktime: Students should work in pairs to complete *Problems 5, 6, and 7a and b.*

(Note: *Problems 7c and d may* be assigned as homework.)

Item 5, parts d and *e* were added to the original DOE task to address MM2A1c. In *part d*, students can use the *intersect* feature of their graphing calculators to determine where the absolute value function, found in *part c*, intersects the line $y = 1.5$. To verify their work algebraically, students may solve the simple absolute value equation obtained by setting the function found in *part c* equal to 1.5 or they may solve equations obtained by using appropriate "pieces" of the piecewise function developed in *part b*. (Using the training model given, Jim should be 1.5 miles from his house at 22.5 minutes and again at 37.5 minutes into his run.)

Item 5, part e asks students to solve an absolute value inequality. Students may justify their work graphically or algebraically using the methods described above for *part d*. (Jim should be within ¾ of a mile of his home when he has run less than 11.25 minutes or more than 48.75 minutes.)

When most students have completed *Problem 5*, have a class discussion of this problem before allowing students to move on to *Problems* 6 and *7*. Make sure students share both graphical and algebraic approaches to *parts d and e.*

(Note: Students solved simple absolute value equations and inequalities in Grade 8 and in Math I but may need some review of these ideas. *Teachers should provide skill practice on these topics.*

Closing: Allow students to share their work. (See GaDOE Teacher Notes.)

Homework: The problems assigned as *Day 3 Homework* are GaDOE original *Problems 6*, *7 c* and 7*d*. See GaDOE Teacher Notes for solutions. These problems should be discussed thoroughly.

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: In addition to terms listed as ***new vocabulary,*** students should preview:

- calculating average rates of change of quadratic functions
- graphing quadratic functions
- using graphing technology to graph functions
- solving simple linear and quadratic equations

Mathematics II
Planning a Race Strategy
Day 3 Student Task

5. Consider a training run where Jim leaves from home, runs for half an hour so that his distance from home is increasing at a constant rate of 4 miles per hour, and then for the next half hour runs so that his distance from home is decreasing at 4 miles per hour.
   a. Draw a graph of the function that models this training run.

   b. Write a formula for the function as a piecewise function.

   c. Write a single formula for this function using transformations of the absolute value function.

   d. Use your results from *part c* and your graphing calculator to help you determine when Jim is 1.5 miles from home. Verify your answer algebraically.

   e. When was Jim within three-quarters of a mile of his house? Justify you answer.

6. Use the definition of absolute value of a real number to write a piecewise rule for the absolute value function $y = |x|$.

7. Graph each of the following functions as a transformation of a function of the form $y = a|x|$, and then write a piecewise rule for it.
   a. $f(x) = 2|x| - 5$
   b. $f(x) = -|x - 3|$

Mathematics II
Planning a Race Strategy
Day 3 Homework

1. The piecewise function *k,* defined below, represents one of Jim's weekday training runs and models his distance from home in kilometers as a function of time *t* since he left home.

$$k(t)=\begin{cases}0.01t^2+1, & 0\le t<10\\ -0.01t^2+0.4t-1, & 10\le t<30\\ 0.01t^2-0.8t+17, & 30\le t\le 40\end{cases}$$

a. Draw a graph of the function *k* on graph paper. You may find it helpful to use what you know about transformations of the function $y=0.01t^2$.

b. State the domain and range, and interpret these for this training run.

c. Does the graph have any lines of symmetry? Explain.

d. What are the intercepts of the graph, if any? Interpret these for this training run.

e. What is the maximum value of the function? Interpret this value in relation to the training run.

2. Graph each of the following functions as a transformation of a function of the form $y=a|x|$, and then write a piecewise rule for it.

a. $f(x)=\frac{1}{2}|x-2|-2$

b. $f(x)=|x+4|+1$

3. Solve each of the following equations.

a. $7=\dfrac{1}{2}|x-2|-2$

b. $|x+4|+1\le 6$

c. $2|x-3|+4\ge 12$

Atlanta Public Schools

Teacher's Curriculum Supplement

Mathematics II: Unit 5

Task 2: Parking Deck Pandemonium

Mathematics II
Task 2: *Parking Deck Pandemonium* **Day 1/3**
(GaDOE TE Problems 1 - 3)

Standard(s): MM2A1. Students will investigate step and piecewise functions, including greatest integer and absolute value functions.

b. Investigate and explain characteristics of a variety of piecewise functions including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, points of discontinuity, intervals over which the function is constant, intervals of increase and decrease, and rates of change.

New vocabulary: step function, greatest integer function, floor function, points of discontinuity

Mathematical concepts/skills:
- modeling real situations using piecewise functions
- relating characteristics of piecewise functions to given contexts
- evaluating and graphing the greatest integer function

Prior knowledge:
- constructing tables of values for given situations
- using the vertical line test to determine whether a graph represents a function
- relating characteristics of a function to a given context

Essential question(s): What is the greatest integer function and how can it be used to model real-world situations?

Suggested materials: graph paper

Warm-up: Post the following:

As you drive through town, looking for a place to park, you notice that Pete's Parking Deck advertises free parking up to the first hour. Then, the cost is $1 for each additional hour or part of an hour.

In other words, if you park at Pete's for 59 minutes and 59 seconds, parking is free; however, if the time shows at exactly 60 minutes, you pay $1. Similarly, if you park for any time from 1 hour up to 2 hours, then you owe $1; but parking for exactly 2 hours costs $2.

Make a table listing some fees for parking at Pete's for positive times that are 5 hours or less. Be sure to include some non-integer values; write these in decimal form.

Opening: Ask a student to draw a table on the board and label it with the appropriate headings. (Headings should be *time in hours* and *parking fee in dollars*.) Call on various students to enter one or more of their ordered pairs in the table. Have a class discussion based on these ordered pairs.

Once all students understand the fee schedule for Pete's Parking Deck, read and discuss the scenario for the task, explaining to students that the **step function** they will explore in this lesson is the **greatest integer function**.

Worktime: Students should work in pairs to complete *Items 1 - 4* of the task.

When students have been given an appropriate amount of time to complete *Item 1*, have a whole class discussion of all parts of this item before moving on to *Item 2*.

Students need to have a clear understanding of the meaning and significance of open and closed endpoints, both in relation to the "pieces" of the graph and in writing the piecewise function using function notation. What would it mean if both endpoints for a given value of x where closed? What would it mean if both were open? (See GaDOE Teacher Notes.)

Have students read the introduction to *Item 2* silently and then call on students to summarize what they have read. Emphasize the definition of the greatest integer function and the notation used to denote it.

Allow students to complete *Items 2 – 4*, monitoring work carefully to be sure that students have graphed the greatest integer function correctly.

Item 4, and the short discussion on discontinuity that precedes the item, were added to the original DOE task for purposes of this Supplement. It is important that students have a simple, intuitive understanding of continuity and points of discontinuity. Students may respond to *Item 4* by saying that points of discontinuity occur for all x where x is an integer, or for all x such that $x \in \{\ldots -3, -2, -1, 0, 1, 2, 3 \ldots\}$.

Closing: Allow students to share their work on *Items 2 – 4*. As students share, allow them to develop an Anchor Chart that contains the graph of the greatest integer function, its domain, range, points of discontinuity, and evaluations for specific values of the domain using all three different notations for the function.

Homework:

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: In addition to terms listed as ***new vocabulary,*** students should preview:

- constructing tables of values for situations similar to those given in the task
- using the vertical line test to determine whether a graph represents a function
- relating characteristics of a function to a given context

Mathematics II
Parking Deck Pandemonium
Day 1 Student Task

In this task, you will explore a particular type of piecewise function called a **step function**. Although there are many different kinds of step functions, two common ones are the **least integer function**, or the "ceiling function," and the **greatest integer function**, sometimes called the "floor function."

The fee schedule at parking decks is often modeled using a step function. Let's look at a few different parking deck rates to see the step functions in action. (Most parking decks have a maximum daily fee. However, for our exploration, we will assume that this maximum does not exist.)

1. As you drive through town, Pete's Parking Deck advertises free parking up to the first hour. Then, the cost is $1 for each additional hour or part of an hour.

   Thus, if you park at Pete's Parking Deck for 59 minutes and 59 seconds, parking is free; however, if the time shows at exactly 60 minutes, you pay $1. Similarly, if you park for any time from 1 hour up to 2 hours, then you owe $1; but parking for exactly 2 hours costs $2.

   a. Make a table listing some fees for parking at Pete's for positive times that are 5 hours or less. Be sure to include some non-integer values; write these in decimal form. Then draw (by hand) the graph that illustrates the fee schedule at Pete's for *x* hours, where $0 < x \leq 5$.

   b. Use your graph to determine the fee if you park for 3 ½ hours. What about 3 hours, 55 minutes? 4 hours? 5 minutes?

   c. What are the *x*- and *y*-intercepts of this graph? What is the interpretation in the context of Pete's Parking Deck?

   d. What do you notice about the time written in decimal form and the corresponding fee? Make a conjecture about the fee if you were to park at Pete's Parking Deck for 10.5 hours (assuming no maximum fee).

   e. Write a piecewise function *P* to model the fee schedule at Pete's Parking Deck.

2. If Pete's Parking Deck allows fees to accumulate for multiple days for a car that is just left in the lot, then, theoretically, there is no maximum fee. Thus, to write the rule for your piecewise function model in *Item 1, part d,* immediately above, the statement of the rule for *P* needed to show a pattern that continues forever.

   There is a useful standard function that gives the same values as the function for the parking fees at Pete's Parking Deck but is defined for negative real numbers as well nonnegative ones. This function is called the **greatest integer function.** The greatest integer function is determined by locating the *greatest integer that is less than or equal to* the *x*-value in question. For any real number $x, [x]$, is used to denote the greatest integer function applied to x.

   a. Evaluate each of the following by determining the greatest integer less than or equal to the *x*-value, that is, let $f(x)=[x]$, where x is any real number.

      i) $f(3.6)$ ii) $f(0.4)$ iii) $f(-0.4)$ iv) $f(-1)$ v) $f(-2.2)$

   b. Draw the graph of the greatest integer function, $f(x)=[x]$, for the viewing window $-10 \le x \le 10$.

   c. What is the domain of the greatest integer function, $f(x)=[x]$?

   d. What is the range of the greatest integer function, $f(x)=[x]$?

   e. What is the shape of the graph beyond the given viewing window? Can you indicate this on your hand-drawn graph?

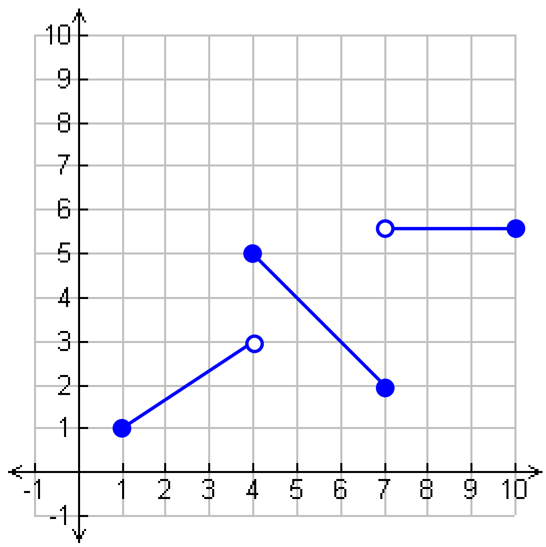
Several different notations are used for the greatest integer function. The two most common are $f(x)=[x]$, which we have used so far, and $f(x)=[\![x]\!]$. However, computer scientists use another name for the greatest integer function; they call it the **floor function**, and use the notation $f(x)=\lfloor x \rfloor$. To help remember this notation, note that the bars on the brackets occur only at the bottom (or floor) of the straight line segments.

The name "floor function" may be more helpful in remembering how the formula for the function works. This function, by whatever name it is called, gives an integer value output that is less than or equal to the value of the input number. Of course, there are many integers less than or equal to any given number, so to make this a function, we choose the largest integer that meets this condition. Choosing the largest integer that is less than or equal to the input number gives us the name "greatest integer function" but that name can be misleading because the output is always **less than or equal to** the input. The name "floor function" should help you remember that the output is less than or equal to the input number just as the height of the floor of a room is less than or equal to the height of any object in the room.

3. Practice working with the various notations for the greatest integer function. For each expression below, rewrite the expression using one of the other notations for the greatest integer function and then evaluate the expression.

i) $[5.3]$ ii) $[\![-4.317]\!]$ iii) $\lfloor 10.1+3.4 \rfloor$ iv) $[\![2.3-5.7]\!]$ v) $\lfloor (1.34)(-6.8) \rfloor$

Most of the functions we have studied to this point, including the piecewise functions investigated in the first task of this unit, can be described as **continuous** functions, meaning they can be drawn with one continuous motion of the pen. This is not the case with the greatest integer function. In order to draw its graph, we must draw a segment, stop, and lift our pencil to draw the next segment. The *x*-values at which we must stop and lift our pencils are called **points of discontinuity**. In the function shown below, points of discontinuity occur at $x = 3$ and $x = 5$.



4. Describe the *x*-values of the points of discontinuity for the greatest integer function for the graph shown above.

Mathematics II
Parking Deck Pandemonium
Day 1 Homework

1. Let f represent the greatest integer function. For each item below, use a different notation to write an expression for $f(x)$ and evaluate that expression.
   a. $f(5.9)$ *b.* $f(-2.1)$ *c.* $f(-4.8)$

2. A well-known mail order company charges shipping based on the total weight of all the items purchased by a customer.

   - The charge to ship items that weigh less than 3 pounds is $5.
   - The charge to ship items that weigh at least 3 pounds but less than 6 pounds is $10.
   - The charge to ship items that weigh at least 6 pounds but less than 9 pounds is $15.
   - The charge to ship items that weigh at least 9 pounds but less than 12 pounds is $20.
   - The charge to ship items that weigh at least 12 pounds but less than 15 pounds is $25.
   - The pattern for charging continues.

   a. Draw a graph that represents the relationship between the total weight of all the items purchased by a customer and his or her shipping charges.
   b. This graph represents a function. Why?
   c. What is the domain of the function? Explain what the domain is in the context of the problem.
   d. What is the range of the function? Explain what the range is in the context of the problem.
   e. At what points is the function discontinuous?
   f. Use function notation to write a piecewise rule that models this situation for items that weigh less than 25 pounds.

Mathematics II
Task 2: *Parking Deck Pandemonium* **Day 2/3**
(GaDOE TE Problems 4 and 5)

Standard(s): MM2A1. Students will investigate step and piecewise functions, including greatest integer and absolute value functions.

c. Investigate and explain characteristics of a variety of piecewise functions including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, points of discontinuity, intervals over which the function is constant, intervals of increase and decrease, and rates of change.

New vocabulary:

Mathematical concepts/skills:

- modeling real situations using piecewise functions
- relating characteristics of piecewise functions to given contexts
- representing and graphing transformations of the greatest integer function

Prior knowledge:

- transformations of functions, including vertical shifts, vertical stretches and shrinks, and reflections across the *x*- and *y*- axes
- relating characteristics of a function to a given context

Essential question(s): How do transformations affect the greatest integer function?

Suggested materials: graph paper

Warm-up: Have students compare homework with a partner. Ask them to be prepared to share their work with the class.

Opening: *Discuss the Day* 1 Homework as a means of setting the stage for this lesson.

Worktime: Students should work in pairs to complete *Items 5 and 6* of the task.

When students have been given an appropriate amount of time to complete *Item 5*, have a whole class discussion of all parts of this item before moving on to *Item 6*.

Closing: Allow students to share their work on *Item 6*.

Homework:

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: In addition to terms listed as ***new vocabulary,*** students should preview:

- transformations of functions, including vertical shifts, vertical stretches and shrinks, and reflections across the *x*- and *y*- axes
- relating characteristics of a function to a given context

Mathematics II
Parking Deck Pandemonium
Day 2 Student Task

5. Penny's Parking Deck is down the street from Pete's. Penny recently renovated her deck to make the parking spaces larger, so she charges more per hour than Pete. Penny's Parking Deck offers free parking up to the first hour (i.e., the first 59 minutes). Then, the cost is $2 for each additional hour or part of an hour. (If you park for 1 ½ hours, you owe $2.)

   a. Draw the graph that illustrates the fee schedule at Penny's Parking Deck for x hours, where $0 < x \le 5$.

   b. How does the graph for Penny's Parking Deck compare with the graph of Pete's Parking Deck (from *Item 1, part a*)? To what graphical transformation does this change correspond?

   c. If you were to form a line by connecting the left endpoints of the steps in the graph for Pete's Parking Deck, found in answering *Item 1, part a,* what would be the equation of the resulting linear function?

   d. If you were to form a line by connecting the left endpoints of the steps in the graph for Penny's Parking Deck, found in answering *part a* for this item (*Item 4*), what would be the equation of the resulting linear function?

   e. How do your answers for *parts c* and *d* relate to your answer to *part b* in this item (*Item 4*)?

   f. Write the function, g, in terms of the greatest integer function, that gives the same values as the function for the parking fees at Penny's Parking Deck but extends the domain to include all real numbers.

   g. Draw the graph of $y = g(x)$ over the domain $-10 \le x \le 10$.

   h. What are the domain and range for the function g?

6. Pablo's Parking Deck is across the street from Penny's deck. Pablo decided not to provide any free parking. Pablo charges $1 for less than an hour, $2 for an hour or more but less than two hours, and so forth, adding $1 whenever the time goes over the next hour mark. (If you park for 59 minutes and 59 seconds, you owe $1; if you park for 1 hour, you owe $2; etc.)

   a. Draw the graph that illustrates the fee schedule at Pablo's Parking Deck for x hours, where $0 < x \le 5$.

b. How does the graph for Pablo's Parking Deck compare with the graph for Pete's Parking Deck? To what graphical transformation does this change correspond?

c. Write the function, h, in terms of the greatest integer function, that gives the same values as the function for the parking fees at Pablo's Parking Deck but extends the domain to include all real numbers. (What are the two different forms that this function could take?)

d. Draw the graph of $y = h(x)$ over the domain $-10 \le x \le 10$.

e. What are the domain and range for the function h?

Mathematics II
Parking Deck Pandemonium
Day 2 Homework

Describe each of the functions below as a transformation of the greatest integer function, graph the function, and then state its domain and range.

1. $f(x) = 3[x - 1]$
2. $f(x) = -[x + 1]$

Mathematics II
Task 2: *Parking Deck Pandemonium* **Day 3/3**
(GaDOE TE Problems 6 - 8)

Standard(s): MM2A1. Students will investigate step and piecewise functions, including greatest integer and absolute value functions.

d. Investigate and explain characteristics of a variety of piecewise functions including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, points of discontinuity, intervals over which the function is constant, intervals of increase and decrease, and rates of change.

New vocabulary: least integer function, ceiling function

Mathematical concepts/skills:

- modeling real situations using piecewise functions
- relating characteristics of piecewise functions to given contexts
- evaluating the least integer function for specific values of x
- graphing the least integer function and related transformations
- using technology to graph the greatest integer and least integer functions

Prior knowledge:

- transformations of functions, including vertical shifts, vertical stretches and shrinks, and reflections across the x- and y- axes
- relating characteristics of a function to a given context

Essential question(s): What is the least integer function and how do I use it to model real-world situations?

Suggested materials: graph paper, graphing calculator

Warm-up: Post the following:

*Use your graphing calculator to check the graphs you drew for homework. The TI-83, and TI-84 families use the expression **int(x)** to represent the greatest integer function. The expression may be found by entering **MATH, NUM, 5**.*

Opening: Once students have had time to check the graphs they drew for *Day 2 Homework*, discuss the assignment. Students should be able to describe transformations of the functions, draw their graphs by hand and with the graphing calculator, and state domains and ranges.

Worktime: Students should work in pairs to complete *Items 7 - 10* of the task.

(Note: All or part of *Item 9* can be assigned as homework. *Item 10* might be used as an extension or enrichment problem.)

When students have been given an appropriate amount of time to complete *Item 7*, have a whole class discussion of this problem before moving on to *Item 8*.

Ask students to read the paragraph between *Items 7* and *8* silently and then call on several students to summarize the information aloud.

As students work on *Item 8*, monitor their work carefully to be sure they understand how to evaluate and graph the least integer function.

Closing: Allow students to share their work on *Items 8 – 10.* As students discuss *Item 8*, allow them to develop an Anchor Chart containing the graph of the least integer function, its domain and range, and evaluations for specific values of the domain using notation for the function.

Homework:

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: In addition to terms listed as ***new vocabulary,*** students should preview:

- transformations of functions, including vertical shifts, vertical stretches and shrinks, and reflections across the *x*- and *y*- axes
- relating characteristics of a function to a given context

Mathematics II
Parking Deck Pandemonium
Day 3 Student Task

7. Padma's Parking Deck is the last deck on the street. To be a bit more competitive, Padma decided to offer parking for each full hour at $1/hour. (If you park for 59 minutes or exactly 1 hour, you owe $1; if you park for up to and including 2 hours, you owe $2.)

   a. Draw the graph that illustrates the fee schedule at Padma's Parking Deck for *x* hours, where $0 < x \leq 5$.

   b. To which of the graphs of the other parking deck rates is the graph for Padma's Parking Deck most similar? How are the graphs similar? How are they different?

The fee schedule at Padma's Parking Deck is modeled by the **least integer function**, or ***ceiling function***. The least integer function is determined by locating the ***least integer that is greater than or equal to*** the *x*-value in question. The least integer function is also called the ceiling function and written with the following notation (analogous to the floor function notation): $c(x) = \lceil x \rceil$. To help remember this notation, note that the bars on the brackets occur only at the top (or ceiling) of the straight line segments.

8. Let $c(x) = \lceil x \rceil$.
   a. Evaluate each of the following by determining the least integer greater than or equal to the *x*-value.
      i) $c(3.5)$ ii) $c(4)$ iii) $c(-2.1)$ iv) $c(-.1)$ v) $c(1.6)$

   b. Draw the graph of $y = c(x)$ over the domain $-10 \leq x \leq 10$.

   c. What are the domain and range for the function *c*?

   d. Suppose Padma chose to offer the first full hour free. After that, patrons would be charged $1 for up through each full hour. What transformation of the least integer function would model this parking fee structure?

9. As additional practice with step functions, graph each of the following.
   For each function, state the parent function (either $f(x) = [x]$ or $g(x) = \lceil x \rceil$) and explain what transformations have been applied to the parent function; state domain, range, and *y*-intercept.

   a. $h(x) = 2[x] - 1$ b. $j(x) = -[x] + 2$ c. $k(x) = \lceil x - 2 \rceil$

10. The TI-calculator does not have an expression for the least integer function. Maria says she can still graph the least integer function on her calculator by entering the following:

$$Y = -\ int\ (-x)$$

Is Maria right? Justify your response by drawing this function first by hand and then using your calculator.

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Mathematics II: Unit 5

Task 3: Functions and Their Inverses

Mathematics II
Task 3: *Functions and Their Inverses* **Day 1/3**
(GaDOE TE Please Tell Me in Dollars and Cents
Problems 1 and 2, adapted)

Standard(s): MM2A5. Students will explore inverses of functions.

d. Use composition to verify that functions are inverses of each other.

New vocabulary: composition of functions, composite function, kelvins

Mathematical concepts/skills:

- writing functions to model real situations
- finding the composition of two or more functions
- converting among degrees Fahrenheit, degrees Celsius, and kelvins

Prior knowledge:

- writing functions to model real situations
- determining cost given a percent of increase or decrease
- converting between degrees Fahrenheit and degrees Celsius

Essential question(s): What is meant by the ***composition*** of two or more functions? Does order matter when I compose two functions?

Suggested materials: graph paper

Warm-up: Post the following:

Angela is buying a pair of jeans that are marked 35% off. If x represents the original price of the jeans, write a function that gives the amount Angela will pay after her discount.

Opening: Discuss the warm-up making sure that all students have some form of the function $f(x) = .65x$.

Have students read the introduction of the task down to *Item 1*. Begin by discussing the critical questions that will be addressed in this task. Then make sure all students understand the information related to the rebate and discounts given for the cell phones.

Worktime: Students should work in pairs to complete *Items 1 – 4* of the task.

(Note: All or part of *Item 4* can be assigned as homework, if time does not allow completion in class. However, any parts of this item completed for homework should be thoroughly discussed in class before students move on to *Item 5*.)

In *Item 1, part a* make sure students understand that *P*(*x*) does not include the 15% student discount. When students have had time to complete *Item 2*, have a whole class discussion before allowing students to begin *Item 3*. Discussion should include the definition and notation for the composition of two functions.

Before students begin *Item 4*, you may want to have a brief discussion of the kelvin temperature scale. A discussion can be found at either of these websites:

http://lamar.colostate.edu/~hillger/temps.htm
http://en.wikipedia.org/wiki/Kelvin

Closing: Allow students to share their work on *Items 3 and 4*
Students may be surprised to find that Sidney and Marcus paid different amounts for their phones. This is a great place to stress that in some cases order matters in the composition of two functions. This will give students an opportunity to more deeply understand that when inverse functions are composed, in either order, the result is still the identity function.

Homework:

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: In addition to terms listed as ***new vocabulary,*** students should preview:

- determining cost given a percent of increase or decrease
- converting between degrees Fahrenheit and degrees Celsius

Mathematics II
Functions and Their Inverses
Day 1 Student Task

Functions and Their Inverses

In this task, we will focus on finding inverses of some of the functions you have studied to this point. In order to understand the ideas related to inverses, we will answer several critical questions:

- What is meant by the ***composition*** of two or more functions?
- What is the relationship between a function and its inverse?
- What procedure do we use for finding the inverse of a given function?
- How do we prove that two functions are inverses of each other?
- How are the graphs of inverse functions related?
- Which functions have inverses?

We will begin with Marcus and his girlfriend Sidney. Both Marcus and Sidney need new cell phones. The cell phone provider near their school is giving a 15% discount on phones for students who bring in a coupon from the school newspaper.

On the Saturday that Sidney and Marcus go to buy their phones they discover that the phone company is also giving an on-the-spot $35 rebate to any customer who buys a touch phone that day.

1. Suppose x represents the original price of a touch phone.
   a. Write a function $P(x)$ that represents how much a customer would pay for a touch phone after the $35 rebate.

   b. Write a function $D(x)$ that represents how much a customer would pay for a touch phone if they got *only* the 15% discount.

2. Sidney knows exactly which phone she wants. She goes into the store, picks up her phone, and goes straight to the sales clerk to pay. After the clerk applies the $35 rebate, Sidney hands her the student coupon for 15% off, which the clerk then applies.

   a. Write a function $H(x)$ that represents how much Sidney paid for her phone.

   b. To determine how much Sidney paid for her phone, the clerk first applied the function $P(x)$, written in *Item 1a* and then applied the result of $P(x)$ to the function $D(x)$ written in *Item 1b*. This process, using the output value from one function as the input of another, is referred to as the ***composition*** of two functions. In this case $H(x)$ is the composition of functions D and P. There are two different notations for compositions of functions. We can write this composition as $(D \circ P)(x)$ or as $D(P(x))$. In other words, $H = (D \circ P)(x) = D(P(x))$.

Suppose the original price of Sidney's phone is $99. Find $P(99)$ and then $D(P(99))$. How much did Sidney pay after the rebate and the 15% discount?

Using the function you wrote in *Item 2a*, find $H(99)$. Did you get the same amount?

3. Marcus has finally chosen a phone. He takes his phone to another sales clerk. Marcus hands the clerk the student coupon which the clerk applies. He then reminds the clerk that this is a touch phone and asks for his $35 rebate.
   a. How does the manner in which Marcus paid for his phone, differ from the way Sidney paid for hers?

   b. Write a function $F(x)$ that represents the amount of money Marcus will pay for his phone.

   c. Represent $F(x)$, using both notations, as the composition of the two functions D and P. How does this composition differ from $H(x)$, the amount Sidney pays?

   d. The original cost of Marcus's phone was also $99. How much did he pay for his phone?

   e. Compare the amounts Marcus and Sidney paid for their phones. With this comparison in mind, what statement might you make about the composition of two functions?

Let's consider one more example of the composition of functions.

4. Aisha needs to make a chart of experimental temperature data for her science project. She has measured all of her temperatures in degrees Fahrenheit but her teacher would like for her to list the data in kelvins.

   a. Aisha found the following function for converting from degrees Fahrenheit to degrees Celsius: $C(x)=\frac{5}{9}(x-32)$, where x represents degrees Fahrenheit and $C(x)$ represents degrees Celsius.

      Use this formula to convert freezing (32°F) and boiling (212°F) to degrees Celsius.

   b. Aisha also found the following function for converting degrees Celsius to kelvins:

      $K(x)=x+273$, where x represents degrees Celsius and $k(x)$ represents kelvins.

      Use this formula and the results of *part a* to express freezing and boiling in kelvins.

   c. Use the formulas from *part a* and *part b* to convert the following to kelvins: – 238°F, 5000°F.

d. Write the function $h(x) = K(c(x))$. What does x represent in this function? What does $h(x)$ represent?

e. Use your function $h(x)$ to convert -238°F and 5000°F to kelvins. Do your results match the results you got in *part c*?

In the examples provided above, you have explored the operation on functions called ***composition of functions.***

Composition of functions is defined as follows: If f and g are functions, the ***composite function*** $f \circ g$ (read this notation as "*f* composed with *g*) is the function with the formula $(f \circ g)(x) = f(g(x))$, where x is in the domain of g and $g(x)$ is in the domain of f.

Mathematics II
Functions and Their Inverses
Day 1 Homework

1. Let $f(x) = 3x + 5$ and $g(x) = x - 4$.
   a. Find $f\left(g\left(x\right)\right)$.
   b. Find $\left(g \circ f\right)\left(x\right)$.

2. Let $h(x) = 3(4x+1)^2$. Find two functions f and g such that $\left(f \circ g\right)\left(x\right) = h\left(x\right)$.

3. The wholesale price of a box of 25 CDs is \$15 (the mark-up price) plus 25 times the manufacturing cost per disk which is represented by x.

   a. Write a function $W(x)$ that gives the wholesale price of a box of 25 CDs in terms of the manufacturing cost per disk.

   b. Retailers get a 20% markup on each box of 25 CDs they sell. Write a function $R(y)$ that gives the retail cost of a box of CDs if y represents the wholesale price of a box of CDs.

   c. Write a function $C(x)$ that gives the cost to the customer of a box of 25 CDs if x represents the manufacturing cost of each CD.

   d. How much will it cost you to buy a box of 25 CDs if the manufacturing cost per disk is \$.55

Mathematics II
Task 3: *Functions and Their Inverses* **Day 2/3**
(GaDOE TE Please Tell Me in Dollars and Cents
Problems 4 - 7, and 9 adapted)

Standard(s): MM2A5. Students will explore inverses of functions.

a. Discuss the characteristics of functions and their inverses, including one-to-oneness, domain, and range.
b. Determine inverses of linear, quadratic, and power functions and functions of the form $f(x)=\frac{a}{x}$, including the use of restricted domains.
d. Use composition to verify that functions are inverses of each other.

New vocabulary: inverse of a function, invertible functions

Mathematical concepts/skills:

- understanding the relationship between inverse functions
- proving that two functions are inverses by verifying that $f^{-1}(f(x))=x$ and $f\left(f^{-1}(x)\right)=x$ for any real number *x*.
- finding the inverse of a given function

Prior knowledge:

- solving equations for one variable in terms of another
- converting from verbal to algebraic expressions
- evaluating functions using function notation

Essential question(s): What is the relationship between inverse functions? How do we prove that two functions are inverses of each other? What procedure do we use for finding the inverse of a given function?

Suggested materials: graph paper

Warm-up: Have students compare homework with a partner. Ask them to be prepared to share their work with the class.

Opening: Discuss the homework, particularly any parts of *Item 4* of the *Day 1 Student Task* that were assigned as homework. Understanding all parts of this item is crucial to success in the remainder of the task.

Begin today's lesson by reading the introduction to *Item 5* and reminding students of the function introduced in *Item 4b:* $K(x)=x+273$, where *x* represents degrees Celsius and *k*(*x*) represents kelvins . Ask students why this function can be written as the equation *K* = *C* + 23.

Worktime: Students should work in pairs to complete *Items 5 – 9* of the task.

Once students have had ample time to complete Item 5, have a whole class discussion of this item and of the definition of inverse functions. There is a concerted effort in this problem to help students see the variable in a function formula as simply an algebraic representation of the input to be operated upon according to the rules of the function. The "letter" used to represent that input value is unimportant. Students should also understand that composing a function and its inverse in either order results in the identity function. (See GaDOE Teacher Notes, page 69.)

(Note: Parts of *Items 7 – 9* not completed in class may be assigned for homework. However, all parts of these items should be thoroughly discussed in class.)

Closing: Allow students to share their work on *Items 6 – 9.* (See GaDOE Teacher Notes, pages 70 – 75)

Homework:

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: In addition to terms listed as ***new vocabulary,*** students should preview:

- solving equations for one variable in terms of another
- converting from verbal to algebraic expressions
- evaluating functions using function notation

Mathematics II
Functions and Their Inverses
Day 2 Student Task

Now we let Aisha, and others, help us answer our next three critical questions:

- What is the relationship between a function and its inverse?
- How do we prove that two functions are inverses of each other?
- What procedure do we use for finding the inverse of a given function?

It turns out that Aisha's project was selected to compete at the science fair for the school district. However, the judges made one suggestion – that Aisha express temperatures in degrees Celsius rather than kelvins. (Aisha really wishes they would make up their minds).

5. Remember that to convert from degrees Celsius to kelvins, we used the function $K(x) = x + 273$, where *x* represents degrees Celsius and *K(x)* represents kelvins. Suppose we write that function as the formula $K = C + 273$.

   a. Find a formula for *C* in terms of *K*. What does this formula tell you?

   b. Write a function *C* such that *C*(*x*) is the Celsius temperature corresponding to a temperature of *x* kelvins.

   c. Explain in words the process for converting from degrees Celsius to kelvins. Do the equation $K = C + 273$ and the function $K(x) = x + 273$ both express this idea?

   d. Explain verbally the process for converting from kelvins to degrees Celsius. Do your formula from *part a* above and your function *c* from *part b* both express this idea?

   e. Calculate the composite function $(C \circ K)(x)$, and simplify your answer. What is the meaning of *x* when we use *x* as input to this function?

In taking the compositions of the functions *K* and *C* in *Items 5e and 5f,* we started with an input number, applied one function, and then used the output from the first function as the input for the other function. In each case, whether we calculated $(C \circ K)(x)$ or $(K \circ C)(x)$ the final output was the starting input number. Your calculations show that this happens for any choice of the input number *x*. Because of this special relationship between *C* and *K*, the function *C* is called the ***inverse of the function* K** *(or vice versa-K is the inverse of C)* **and we use the notation K^{-1}** (read this as "*K* inverse") as another name for the function *C*.

The precise **definition for inverse functions** is: If *f* and *h* are two functions such that

$$(h \circ f)(x) = h(f(x)) = x \text{ for each input } x \text{ in the domain of } f,$$

and

$$(f \circ h)(x) = f(h(x)) = x \text{ for each input } x \text{ in the domain of } h,$$

then *h* is the inverse of the function *f*, and we write $h = f^{-1}$. Also, f is the inverse of the function *h*, and we can write $f = h^{-1}$.

Note that the notation for inverse functions looks like the notation for reciprocals, but in the inverse function notation, the exponent of "–1 " does **not** indicate a reciprocal.

6. Each of the following describes the action of a function *f* on any real number input. For each part, describe in words the action of the inverse function, f^{-1}, on any real number input. Remember that the composite action of the two functions should get us back to the original input.
   a. Action of the function *f* : subtract ten from each input
      Action of the function f^{-1}:
   b. Action of the function *f* : add two-thirds to each input
      Action of the function f^{-1}:
   c. Action of the function *f* : multiply each input by one-half
      Action of the function f^{-1}:
   d. Action of the function *f* : multiply each input by three-fifths and add eight
      Action of the function f^{-1}:

7. For each part of *Item 6* above, write an algebraic rule for the function and then verify that the rules give the correct inverse relationship by showing that $f^{-1}(f(x)) = x$ and $f\left(f^{-1}(x)\right) = x$ for any real number *x*.

Before proceeding any further, we need to point out that there are many functions that do not have an inverse function. We'll learn how to test functions to see if they have an inverse later in this task. For the moment we will focus on functions that have inverses. A function that has an inverse function is called ***invertible.***

8. The tables below give selected values for a function *f* and its inverse function f^{-1}.
   a. Use the given values and the definition of inverse function to complete both tables.

x	$f(x)$
	11
3	9
	7
10	
15	3

x	$f^{-1}(x)$
3	
5	10
7	6
	3
11	1

   b. For any point (*a, b*) on the graph of *f*, what is the corresponding point on the graph of f^{-1}?

   c. For any point (*b, a*) on the graph of f^{-1}, what is the corresponding point on the graph of f? Justify your answer.

As you have seen in working through *Item 8*, if *f* is an invertible function and *a* is the input for function *f* that gives *b* as output, then *b* is the input to the function f^{-1} that gives *a* as output. Conversely, if *f* is an invertible function and *b* is the input to the function f^{-1} that gives *a* as output, then *a* is the input for function f that gives *b* as output. Stated more formally with function notation we have the following property:

Inverse Function Property: For any invertible function *f* and any real numbers *a* and *b* in the domain and range of *f*, respectively,

$$f(a) = b \text{ if and only if } f^{-1}(b) = a.$$

9. After Aisha had converted the temperatures in the scientific journal article from kelvins to Celsius, she decided, just for her own information, to calculate the corresponding Fahrenheit temperature for each Celsius temperature.
   a. Use the formula $C = \frac{5}{9}(F - 32)$ to find a formula for converting temperatures in the other direction, from a temperature in degrees Celsius to the corresponding temperature in degrees Fahrenheit.

   b. Now let $C(x) = \frac{5}{9}(x - 32)$, be the function that gives the temperature in degrees Celsius for any given temperature of *x* degrees Fahrenheit. What would the inverse of the function *C* give? Use the formula you found in *part a* to help you find a formula for $C^{-1}(x)$.

   c. Check that, for the functions $C(x)$ and $C^{-1}(x)$ from *part b*, $C\left(C^{-1}(x)\right) = C^{-1}\left(C(x)\right) = x$ for any real number *x*.

Mathematics II
Functions and Their Inverses
Day 2 Homework

The cost of having carpet installed is $75 for delivery and $14 per square foot.

1. Write a function $f(x)$ that gives the cost of having x square feet of carpet installed.

2. Find $f^{-1}(x)$. What does $f^{-1}(x)$ give you?

3. Verify that the functions you wrote in *Items 1* and *2* are inverses.

Mathematics II
Task 3: *Functions and Their Inverses* **Day 3/3**
(GaDOE TE Please Tell Me in Dollars and Cents
Items 13 and 14, adapted)

Standard(s): MM2A5. Students will explore inverses of functions.

a. Discuss the characteristics of functions and their inverses, including one-to-oneness, domain, and range.
b. Determine inverses of linear, quadratic, and power functions and functions of the form $f(x) = \frac{a}{x}$, including the use of restricted domains.
c. Explore the graphs of functions and their inverses.
d. Use composition to verify that functions are inverses of each other.

New vocabulary:

Mathematical concepts/skills:

- understanding the relationship between inverse functions
- proving that two functions are inverses by verifying that $f^{-1}(f(x)) = x$ and $f\left(f^{-1}(x)\right) = x$ for any real number *x*.
- finding the inverse of a given function
- comparing the domains and ranges of functions and their inverses
- determining the relationship between the graph of a function and its inverse
- determining which functions are invertible

Prior knowledge:

- solving equations for one variable in terms of another
- converting from verbal to algebraic expressions
- evaluating functions using function notation
- relating characteristics of a function to a given situation
- graphing transformations of the six basic functions
- using the vertical line test to determine whether a graph represents a function

Essential question(s): What procedure do we use for finding the inverse of a given function? How are the graphs of inverse functions related? Which functions have inverses?

Suggested materials: graph paper

Warm-up: Have students compare homework with a partner. Ask them to be prepared to share their work with the class.

Opening: Discuss homework, particularly any parts of *Items 7 – 9* of the *Day 2 Student Task* that were assigned as homework.

Worktime: Students should work in pairs to complete *Items 10 - 14* of the task.

Once students have had ample time to complete *Item 10*, have a whole class discussion of this item. *Items d – f*, related to domain and range, should be thoroughly discussed thoroughly.

Closing: Allow students to share their work on *Items 11 – 14.*

Homework:

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: In addition to terms listed as ***new vocabulary,*** students should preview:

- solving equations for one variable in terms of another
- converting from verbal to algebraic expressions
- evaluating functions using function notation
- relating characteristics of a function to a given situation
- graphing transformations of the six basic functions
- using the vertical line test to determine whether a graph represents a function

Mathematics II
Functions and Their Inverses
Day 3 Student Task

Item 9 illustrates the general algebraic process for finding the formula for the inverse function when we are given the formula for the original function. This process focuses on the idea that we usually represent functions using *x* for inputs and *y* for outputs and applies the inverse function property.

Let's look at one more example.

10. Suppose Molly charges a fixed rate of \$10 plus 9 dollars per hour for every babysitting job. The function $f(x) = 9x + 10$ gives the amount of money Molly would make for *x* hours of babysitting.
    a. What would the inverse of this function tell us?

    b. To find $f^{-1}(x)$, we write the function $f(x)$ as $y = 9x + 10$ and solve for *x*. This gives us, $x = \frac{y-10}{9}$. The inverse function property tells us that the output of f becomes the input of f^{-1}. Therefore, we write the equation $x = \frac{y-10}{9}$ as the inverse of f using *x* to represent our input and $f^{-1}(x)$ or *y* to represent our output.

    $$f^{-1}(x) = \frac{y-10}{9}$$

    What does *x* represent in this function? What does $f^{-1}(x)$ or *y* represent?

    c. Find $f^{-1}(46)$ and explain what it means.

    d. State the domain and range for the function $f(x)$ that models Molly's pricing schedule for babysitting.

    e. State the domain and range for $f^{-1}(x)$ in the context of Molly's pricing schedule.

    f. In general, what are the relationships between the domains and ranges of an invertible function and its inverse? Explain your reasoning.

    g. Remember that we prove that two functions are inverses by verifying that $f\left(f^{-1}(x)\right) = f^{-1}\left(f(x)\right) = x$. Prove that $f(x)$ and $f^{-1}(x)$ in this item are inverses.

In the next few items we explore another of our critical questions: How are the graphs of inverse functions related?

11. For each part below, use a standard, square graphing window with $-10 \le x \le 10$ and $-10 \le y \le 10$.

   a. For functions in *Item 6*, *part a,* graph *f*, f^{-1}, and the line $y = x$ on the same axes.

   b. For functions in *Item 6*, *part c,* graph *f*, f^{-1}, and the line $y = x$ on the same axes.

   c. For functions in *Item 6*, *part d,* graph *f*, f^{-1}, and the line $y = x$ on the same axes.

   d. If the graphs were drawn on paper and the paper were folded along the line $y = x$, what would happen?

   e. Do you think that you would get the same result for the graph of any function *f* and its inverse when they are drawn on the same axes using the same scale on both axes? Explain your reasoning.

12. Consider the function $f(x) = \frac{-3}{x}$.

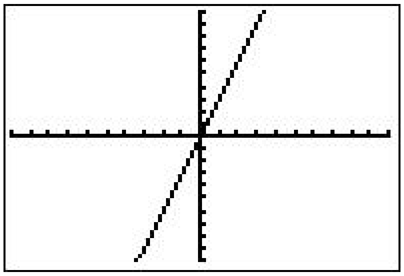
   a. Find the inverse function algebraically.

   b. Draw an accurate graph of the function *f* on graph paper and use the same scale on both axes.

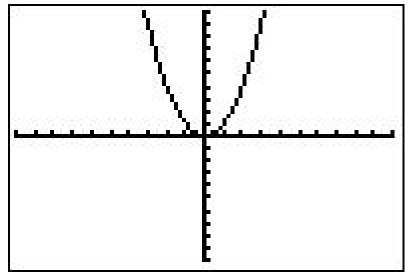
   c. What happens when you fold the paper along the line $y = x$? Why does this happen?

Finally, let's look at our last question: Which functions have inverses and which do not?

To investigate this question, we will compare two very simple relations $y = 3x$ and $y = x^2$. We remember that for any relation to be a function, it must be true that for every element in the domain, there is paired exactly one element in the range. Looking at the graphs of these functions, and using the vertical line test, we can quickly see that both of these relations are also functions: for every input, there is exactly one output.



$y = 3x$



$y = x^2$

Table 1: $f(x) = 3x$	
x	$3x$
3	9
2	6
1	3
0	0
-1	-3
-2	-6
-3	-9

Table 2: $g(x) = x^2$	
x	x^2
3	9
2	4
1	3
0	0
-1	-3
-2	4
-3	-9

13. Let $f(x) = 3x$ *and* $g(x) = x^2$.

a. Use the values in *Table 1* and what you have learned so far to make a table of values that would correspond to those of an inverse relation for *f*.

b. On the same coordinate axes as $f(x)$ above, draw a graph that would correspond to that of an inverse relation for *f*.

c. Do the ordered pairs in your table from *part a* and the graph in *part b* represent a function. Justify your answer.

d. Use the values in *Table 2* to make a table of values that would correspond to those of an inverse relation for *g*.

e. On the same coordinate axes as $g(x)$ above, draw a graph that would correspond to that of an inverse relation for *g*.

f. Do the ordered pairs in your table from *part d* and the graph in *part e* represent a function? Justify your answer.

From your responses in *Item 13*, you may have noticed that even though f and g are both functions (for every input, there is one and only one output) there is a significant difference in the pairings of these two functions. In the case of $f(x) = 3x$, not only is it true that for every input there is exactly one output, it is also true that *for every output there is exactly one input*. We say that there is a **one-to-one correspondence** between the elements in the domain and the elements in the range of this function. Functions for which there is a one-to-one correspondence between the elements of the domain and the elements of the range are said to be **one-to-one functions**.

14. One-to-one functions can be recognized very quickly using a horizontal line test.
    a. Explain how you might use a horizontal line test to determine whether a function is one-to-one. Why does this work?

    b. Is $g(x) = x^2$ a one-to-one function? Justify your answer.

    c. From your experience in *Item 13*, what functions do you think have inverses that are also functions? Justify your answer.

ATLANTA PUBLIC SCHOOLS
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Mathematics II: Unit 5

Task 4: Who Wants to Be a Millionaire?

Mathematics II
Task 4: ***Who Wants to Be a Millionaire?*** **Day 1/2**

Standard(s): MM2A2. Students will explore exponential functions.

a. Extend properties of exponents to include all integer exponents.
b. Investigate and explain characteristics of exponential functions, including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, rates of change, and end behavior.
e. Understand and use basic exponential functions as models of real phenomena.

New vocabulary: exponential function, exponential growth function, exponential decay function

Mathematical concepts/skills:

- writing exponential functions to model real-world situations
- relating characteristics of exponential functions to given contexts
- simplifying exponential expressions involving integer exponents

Prior knowledge:

- writing functions to model real-world situations
- relating characteristics of a function to a given situation
- simplifying exponential expressions involving integer exponents

Essential question(s): What kinds of real situations are modeled by exponential functions?

Suggested materials: graph paper

Warm-up: Post the following:

Simplify the following expression:

$$\frac{\left(x^2\right)^3 y^4}{2y}$$

Opening: Discuss the warm-up as a means of reviewing properties of exponents. Students should have simplified expressions involving positive integer exponents in Grade 8 and in Math I.

Have students read the scenario for the task silently and then allow them to share their thoughts about which payment option they would choose.

Worktime: Students should work in pairs to complete *Items 1 - 5* of the task.

When students have had ample time to complete *Item 1*, have a whole class discussion of this item and of the definition of exponential functions. Ask students why they think functions in the form $f(x) = a \cdot b^x$ are called "exponential functions". Realizing that the functions are so named because the variable is the exponent will help students differentiate exponential functions from power functions (functions in the form $f(x) = ax^n$).

Closing: Allow students to share their work on *Items 2 – 4*

Students should understand several ideas from *Item 4*.

- b^{-n} is defined to be $\frac{1}{b^n}$. In this case, b cannot be 0.
- b^0 is 1 for any real number b. If you have the expression $\frac{b^5}{b^5}$, b cannot be zero because it would cause the denominator to be undefined. However, if you are looking specifically at the expression 0^0, the value would be 1. It may be good for students to know that mathematicians disagree on the value of 0^0. The following is an excerpt from the Math Forum at http://mathforum.org/dr.math/faq/faq.0.to.0.power.html

  The discussion of 0^0 is very old. Euler argues for 0^0 = 1 since a^0 = 1 for a not equal to 0 . The controversy raged throughout the nineteenth century, but was mainly conducted in the pages of the lesser journals: Grunert's Archiv and Schlomilch's Zeitshrift. Consensus has recently been built around setting the value of 0^0 = 1.

An Anchor Chart containing the properties of exponents and definitions of b^{-n} and b^0 may be useful.

Any parts of *Item 5* not completed in class, may be finished for homework. However, teachers should check to be sure that students can apply the properties of exponents.

Homework:

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: In addition to terms listed as ***new vocabulary,*** students should preview:

- simplifying exponential expressions involving integer exponents

Mathematics II
Who Wants to Be a Millionaire?
Day 1 Student Task
Day 1 Student Task

It seems that you have honed your math skills and gained a reputation for expertise in both math and computer science. Apple would like to hire you for 30 days as a troubleshooter on the new iPad project.

The company offers you the following payment options:

Option 1: two cents on the first day, four cents on the second day, and double your salary every day thereafter for the thirty days
Option 2: exactly $1,000,000. (That's one million dollars!)

Which option should you take?

1. Let's begin by investigating your first option.
   a. Complete a table like the one started below. (For now, we will leave the amount of money paid each day in cents.)

Day (t)	1	2	3	4	5	6			
Amount paid on day t (y)	2	4							

   b. How much money will you be paid on day 7? Plot the points representing the amount you will be paid on each day of Week 1.

   c. Write an equation representing the relationship between the day number t and the amount y you will be paid on day t.

The amount of money you will be paid each day, if you choose *Option 1*, can be modeled by an **exponential** function. Exponential functions are a family of functions different from those you have studied before. They have the form

$$f(x) = a \cdot b^x$$

where a can be any real number other than 0, and b can be any positive real number other than 1. If b is greater than 1, the function is called an exponential **growth** function. If b is less than one, the function is often referred to as an exponential **decay** function. Why do you think functions in the form shown above are called *exponential* functions?

2. The equation you wrote in *Item 1, part c* is an exponential function.

a. Write your equation from *Item 1, part c* as the function $f(t)$. What are the values of a and b in this function?

b. In the context of this situation, what is the domain for $f(t)$?

c. Use your function and your calculator to determine how much money you would make on day 30. (Wow!)

d. What is the range of this function?

e. In *Item 1, part b*, you drew a part of the graph of this function. Would it make sense to connect the dots on the graph? Why or why not?

In order to understand and evaluate exponential functions, we need to review the properties of exponents learned in previous grades and possibly introduce some new definitions. We will do that now.

In Grade 8, you were introduced to the following 5 properties of exponents:

- $a^m a^n = a^{m+n}$
- $(a^m)^n = a^{mn}$
- $(ab)^n = a^n b^n$
- $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$, where $b \neq 0$
- $\frac{a^m}{a^n} = a^{m-n}$, where $a \neq 0$

3. For each of the properties above, describe the property in words and give an algebraic example with specific values for m and n to illustrate your work.

In Grade 8 and in Math I, your work focused mainly on *positive*, *integer* exponents. In the next item, we will extend our understanding of exponents to *all* integer exponents. (An exponent may actually be any real number but we will save the discussion of exponents, other than integer exponents, for Math III.)

Since we have already studied positive integer exponents, we need only examine two situations:

- What happens when a real number is raised to the 0 power? In other words, how might we evaluate the expression b^0, when b represents a real number?
- What happens when a real number is raised to a negative power? How do we evaluate b^{-n}, when n is a positive integer and b is a real number?

4. In the case of these two expressions, b^0 and b^{-n}, their values are definitions. Mathematicians simply defined their values in order to make the 5 properties above "work out". Imagine that! See if you can figure out how mathematicians defined these two expressions.

   a. Use the expression $\frac{b^5}{b^5}$ and one of the properties of exponents above to decide how you think b^0 should be defined. Do you think there should be any restrictions on b?

   b. Use the expression $\frac{b^2}{b^5}$ and one of the properties of exponents above to decide how you think b^{-n} should be defined. Do you think there should be any restrictions on b?

5. Use what you have reviewed in *Item 3* and learned in *Item 4* to simplify each of the following expressions. Write each expression in simplest form with no negative exponents.

   a. 6^0

   b. $\frac{x^{-a}}{x^5}$

   c. $2x^2 \bullet 3x^{-2}$

   d. $\left(\frac{x}{y}\right)^2 \bullet \frac{x^3}{y^{-4}}$

   e. $\frac{\left(xy^{-2}\right)^3}{x^2y}$

Mathematics II
Who Wants to Be a Millionaire?
Day 1 Homework

The spread of a rumor or the spread of a disease can often be modeled by an exponential function.

Linda is opening her own ice cream store. She decides to spread the “rumor”of her opening by telling two people about her store each day and asking each person to also tell two other people each day.

1. Let x represent the day number and let y be the number of people who know about the opening on day x. Consider the day before Linda told anyone to be Day 0, so that Linda is the only person who knows about the opening on Day 0. Day 1 is the first day that Linda told someone else about the opening.

   a. Complete the following table.

Day	0	1	2	3	4	5
Number of people who know	1	3				

   b. Graph the points from the table in part a.

2. Write an equation that describes the relationship between x (day) and y (number of people who know) for the situation of spreading the news about the opening of Linda’s ice cream store. What are the values of a and b in this case?

3. How long would it take for at least 500 people to find out about the opening if the rumor spread at this rate?

Mathematics II
Task 4: ***Who Wants to Be a Millionaire?*** **Day 2/2**

Standard(s): MM2A2. Students will explore exponential functions.

a. Extend properties of exponents to include all integer exponents.
b. Investigate and explain characteristics of exponential functions, including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, rates of change, and end behavior.
c. Graph functions as transformations of $f(x) = a^x$.
d. Solve simple exponential equations and inequalities analytically, graphically, and by using appropriate technology.
e. Understand and use basic exponential functions as models of real phenomena.
f. Understand and recognize geometric sequences as exponential functions with domains that are whole numbers.
g. Interpret the constant ratio in a geometric sequence as the base of the associated exponential function.

New vocabulary: geometric sequence, common ratio

Mathematical concepts/skills:

- writing exponential functions to model real-world situations
- determining characteristics of exponential functions, including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, rates of change, and end behavior.
- relating characteristics of exponential functions to given contexts
- simplifying exponential expressions involving integer exponents
- graphing transformations of $f(x) = b^x$
- writing geometric sequences as exponential functions with whole number domains
- solving simple exponential equations

Prior knowledge:

- writing functions to model a real-world situation
- relating characteristics of a function to a given situation
- simplifying exponential expressions involving integer exponents
- graphing transformations of basic functions
- writing sequences as functions with whole number domains

Essential question(s): What kinds of real situations are modeled by exponential functions? What are geometric sequences and how are they related to exponential functions?

Suggested materials: graph paper

Warm-up: Post the following:

Complete the following table:

x	– 2	– 1	0	1	2
3^x					

Opening: Discuss the warm-up as a means of introducing *Item 6* of the Student Task.

Worktime: Students should work in pairs to complete *Items 6 - 9* of the task.

Monitor student work carefully to be sure students are graphing the exponential function $f(x) = 3^x$ correctly.

When students have had ample time to complete *Items 6* and *7* have a whole class discussion of these items. The solution for *Item 7c* is the function $g(t) = 2^{(t+1)} - 2$. The graph of $g(t)$ is a horizontal shift of $f(t)$ one unit to the left and a vertical shift down two units.

Thoroughly discuss the characteristics of exponential functions, including domain, range, asymptotes, zeros, intercepts, intervals of increase and decrease, rates of change, and end behavior.

Closing: Allow students to share their work on *Items 8 and 9.*

Students should understand the definition of geometric sequences and the relationship between exponential functions and geometric sequences. They should also be able to solve simple exponential equations. A lengthy study of geometric sequences, which in the past has included geometric series is **not** necessary in Math II.

One Anchor Chart containing examples of exponential functions and their characteristics and another on geometric sequences may be useful to students.

Homework: It is important that students complete the homework assignment following the Day 2 Student Task and that teachers assess student understanding of these items.

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: In addition to terms listed as ***new vocabulary,*** students should preview:

- simplifying exponential expressions involving integer exponents
- graphing transformations of basic functions
- writing sequences as functions with whole number domains

Mathematics II
Who Wants to Be a Millionaire?
Day 2 Student Task

Now that we understand a little more about integer exponents, let's look again at an exponential function.

6. Consider the function $f(x) = 3^x$.

a. Make a table of values for the function. Include values that occur when x is a positive integer, a negative integer, and 0.

b. Graph the function by plotting your points and drawing a smooth curve through the points. Remember that $f(x) = 3^x$ is defined for all real numbers x, even though we will not study rational and irrational exponents until later.

c. What is the domain of this function? What is the range?

d. Give any intercepts of the graph.

e. Over what intervals is the graph increasing? decreasing?

f. All exponential functions have a horizontal asymptote. Where is the horizontal asymptote for this function? Why? What is the equation of this asymptote?

g. How would you describe the rate of change of this function as x gets large?

7. Let's return to your offer from Apple.

a. If you choose the first option given to you by Apple, how much money would you make over the entire 30 days. To investigate this question, add a third row to the table you started in *Item 1*. Complete this row for at least the first 7 days of work.

Day (t)	1	2	3	4	5	6			
Amount paid on Day t	2	4							
Total amount earned at the end of Day t									

b. Write a function $g(t)$ to represent the total amount of money you will have earned at the end of day t.

c. Predict how the graph of $g(t)$ is related to the graph of $f(t)$, the amount you will make on Day t. Explain your reasoning.

d. Extend the domains of $f(t)$ and $g(t)$ to the set of all real numbers and graph both functions on the same set of axes.

e. Discuss the characteristics of $g(t)$ including domain, range, intercepts, asymptote, intervals of increase and decrease, and rate of change.

We will use the function $f(t)$ representing the amount of money you will earn from Apple on Day t, to discuss one more Math II topic.

In Grade 8, you were introduced to sequences-arithmetic sequences in particular. In Math I, you used function notation to write sequences as functions with domains that are whole numbers. In Unit I of Math II, you investigated arithmetic series. We will now examine another type of sequence called a **geometric sequence**.

8. Suppose you were to write the amounts of money you make on Day t as a list of terms separated by commas. Your list would be the following sequence:

   2, 4, 8, 16, ….

   a. How do you obtain the next term in the sequence from the previous term? Write a recursive formula for this sequence.

   b. Write an explicit formula for this sequence. How does your explicit formula for the sequence compare to your function $f(t)$?

Sequences created by multiplying the previous term by a constant number to obtain the next term are called **geometric** sequences. The constant by which each term is multiplied is referred to as the **common ratio** of the sequence.

9. Consider the sequence 5, 15, 45, 135, …..

   a. Write a recursive formula for this sequence.

   b. Write an explicit formula for the sequence. Hint: The terms of the sequence can be written in the following manner:

   $5 = 5$
   $15 = 5 \cdot 3$
   $45 = 5 \cdot 3 \cdot 3$
   $135 = 5 \cdot 3 \cdot 3 \cdot 3$

   c. Represent the sequence as an exponential function. (Do not forget the domain.) Given that an exponential function is written in the form $f(x) = a \cdot b^x$, what are the values of a and b for this function?

   d. Which term in the sequence has a value of 10,935? Show how you know.

   e. Use what you have learned in this item to write a formula for the n^{th} term of any geometric sequence with a first term a and a common ratio of r.

Mathematics II
Who Wants to Be a Millionaire?
Day 2 Homework

1. For each of the functions below:
   - describe the graph of the function as a transformation of a graph in the form $f(x) = b^x$;
   - graph the function; and
   - give the domain, range, intercepts, equation of the asymptote, intervals of increase, intervals of decrease, and describe the end behavior.

   a. $g(x) = 2 \cdot 3^x$

   b. $h(x) = 2^{-x}$

   c. $k(x) = -\ 2^x + 1$

2. For the sequence given below:

   a. Write the sequence as an exponential function. Include the domain of the function.

   b. Which term of the sequence has a value of -8192? Show how you know.

   $-2, -8, -32, -128, \ldots..$

3. Solve each of the following exponential equations algebraically.

   a. $8^x = 4^3$

   b. $3^{2x} - 81 = 0$

   c. $2 \cdot 5^x = 250$

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Mathematics II: Unit 5

Task 5: The Beginning of a Business

Mathematics II
Task 5: *The Beginning of a Business* **Day 1/1**

Standard(s): MM2A2. Students will explore exponential functions.

d. Solve simple exponential equations and inequalities analytically, graphically, and by using appropriate technology.
e. Understand and use basic exponential functions as models of real phenomena.

New vocabulary: principal, interest, compounding

Mathematical concepts/skills:

- using the standard compound interest formula
- writing exponential functions to model real-world situations
- solving simple exponential equations and inequalities using technology

Prior knowledge:

- writing functions to model a real-world situation
- relating characteristics of a function to a given situation
- simplifying exponential expressions involving integer exponents
- solving simple equations and inequalities using the graphing calculator

Essential question(s): How much money might I earn by investing over a period of time?

Suggested materials: graph paper, graphing calculator

Warm-up: Post the following:

Use your calculator to evaluate the following expression:

$$500\left(1+\frac{0.04}{4}\right)^8$$

Opening: Discuss the warm-up, making sure that all students understand how to enter the expression using the appropriate order of operations. (The value of the expression should be 541.4283523).

Ask students to read the scenario silently, making notes in the margin of all pertinent information. Discuss the scenario. Terminology, including principal, interest, interest rate, and compounding should be discussed. Make sure students understand what is represented by each of the variables involved. Students may also need to be reminded that a rate r must be written as a decimal. (See GaDOE TE, pages 91 – 93.)

Verify the first row of the table and determine entries in the second row as a class to be sure that students understand the calculations.

Worktime: Students should work in pairs to complete *Items 1 and 2* of the task.

Monitor student work carefully to be sure students are obtaining the correct entries for the table.

When students have had ample time to complete *Item 1* have a whole class discussion of this item.

Item 2b may be a bit difficult for students. Ask guiding questions to help them apply properties of exponents to obtain $(1.0075)^{4t} = \left((1.0075)^4\right)^t = (1.030339191)^t$. (See GaDOE TE, page 97).

Items 2d and *2e* have been added to the original DOE task. These items give students opportunities to solve simple exponential equations and inequalities using technology. The *table* and *intersect* features of the graphing calculator are useful in finding solutions for these items.

Closing: Allow students to share their work on *Item 2.*

Homework: The homework assigned for this task includes the following GaDOE items:
Growing by Leaps and Bounds
Part 3, Item 2 and *Part 4, Item 1*

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: In addition to terms listed as ***new vocabulary,*** students should preview:

- simplifying exponential expressions involving integer exponents
- using the *intersect* and *table* features of the graphing calculator to solve simple exponential equations and inequalities

Mathematics II
The Beginning of a Business
Day 1 Student Task

Linda has saved enough money to buy the franchise to an ice cream store. How did she do it? Her mom used to say, "That Linda, why she could squeeze a quarter out of a nickel!" The truth is that Linda learned early in life that patience with money is a good thing. When she was just about 9 years old, she asked her dad if she could put her money in the bank. He took her to the bank and she opened her very first savings account.

Each year until Linda was 16, she deposited her birthday money into her savings account. Her grandparents (both sets) and her parents each gave her money for her birthday that was equal to twice her age; so on her ninth birthday, she deposited $54 ($18 from each couple).

Linda's bank paid her 3% interest, compounded quarterly. The bank calculated her interest using the following standard formula:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where A = final amount, P = principal amount, r = interest rate, n = number of times per year the interest is compounded, and t is the number of years the money is left in the account.

1. Verify the first entry in the following chart, and then complete the chart to calculate how much money Linda had on her 16^{th} birthday. Do not round answers until the end of the computation, then give the final amount rounded to the nearest cent.

Age	Birthday $	Amt from previous year plus Birthday	Total at year end
9	54	0	55.63831630
10			
11			
12			
⋮	⋮	⋮	⋮

2. The formula you used to find the amount of money in Linda's bank account when she was 10 years old can be considered an application of an exponential function where the number of years, t, is the independent variable and the amount of money in the account at the end of t is the dependent variable.

   a. Write the equation for this exponential function.

   b. What are the values of a and b so that it fits the definition of an exponential function?

c. If Linda had not added money to the account each year, how much would she have had in the account at age 16 from her original investment at age 9?
d. Use your graphing calculator to graph this exponential function. Using this function, how much money would Linda have had in her account on her 12^{th} birthday?

e. How long would it have taken for the amount in Linda's account to reach $60?

Mathematics II
The Beginning of a Business
Day 1 Homework
GaDOE *Growing by Leaps and Bound*
Part 3, Item 2 and Part 4, Item 1

1. On her 16th birthday, Linda asked her parents if she could invest in the stock market. She studied the newspaper, talked to her economics teacher, researched a few companies and finally settled on the stock she wanted. She invested all of her money in the stock and promptly forgot about it. When she graduated from college on her 22nd birthday, she received a statement from her stocks and realized that her stock had appreciated an average of 10% per year. How much was her stock worth on her 22nd birthday?

2. The formula you used to find the value of Linda's stocks on her 22nd birthday can be considered an application of an exponential function. Think of the values of P, r, and n as constant and let the number of years vary so that the number of years is the independent variable and the value of the stocks after t years is the dependent variable.

   a. Write the equation for this exponential function.

   b. What are the values of a and b so that it fits the definition of exponential function?

   c. What point on the graph of this function did you find when you calculated the value of Linda's stock at age 22?

   d. Use your graphing calculator to graph this function. Use your graph to find out how much Linda's stock was worth on her 20th birthday.

   e. Approximately when was Linda's stock worth $800?