Atlanta Public Schools

Teacher's Curriculum Supplement

Mathematics II: Unit 1

This document has been made possible by funding from the GE Foundation Developing Futures grant, in partnership with Atlanta Public Schools. It is derived from the Georgia Department of Education Math II Framework and includes contributions from Georgia teachers. It is intended to serve as a companion to the GA DOE Math II Framework Teacher Edition. Permission to copy for educational purposes is granted and no portion may be reproduced for sale or profit.

Preface

We are pleased to provide this supplement to the Georgia Department of Education's Mathematics II Framework. It has been written in the hope that it will assist teachers in the planning and delivery of the new curriculum, particularly in this first years of implementation. This document should be used with the following considerations.

- The importance of working the tasks used in these lessons cannot be overstated. In planning for the teaching of the Georgia Performance Standards in Mathematics teachers should work the task, read the teacher notes provided in the Georgia Department of Education's Mathematics II Framework Teacher Edition, and *then* examine the lessons provided here.

- This guide provides day-by-day lesson plans. While a detailed scope and sequence and established lessons may help in the implementation of a new and more rigorous curriculum, it is hoped that teachers will assess their students informally on an on-going basis and use the results of these assessments to determine (or modify) what happens in the classroom from one day to the next. Planning based on student need is much more effective than following a pre-determined timeline.

- It is important to remember that the Georgia Performance Standards provide a balance of concepts, skills, and problem solving. Although this document is primarily based on the tasks of the Framework, we have attempted to help teachers achieve this all important balance by embedding necessary skills in the lessons and including skills in specific or suggested homework assignments. The teachers and writers who developed these lessons, however, are not in your classrooms. It is incumbent upon the classroom teacher to assess the skill level of students on every topic addressed in the standards and provide the opportunities needed to master those skills.

- In most of the lesson templates, the sections labeled *Differentiated support/enrichment* have been left blank. This is a result of several factors, the most significant of which was time. It is hoped that as teachers use these lessons, they will contribute their own ideas, not only in the areas of differentiation and enrichment, but in other areas as well. Materials and resources abound that can be used to contribute to the teaching of the standards.

On the topic of differentiation, it is critical to reiterate that many of the strategies used in a standards-based mathematics classroom promote differentiation. These strategies include

- the use of rich tasks with multiple points of entry and more than one path to a solution,
- flexible grouping of students,
- multiple representations of mathematical concepts,
- writing in mathematics,
- monitoring of progress through on-going informal and formative assessments, and
- analysis of student work.

We hope that teachers will incorporate these strategies in each and every lesson. It is hoped that you find this document useful as we strive to raise the mathematics achievement of all students in our district and state. Comments, questions, and suggestions for inclusions related to the document may be emailed to Dr. Dottie Whitlow, Executive Director, Mathematics and Science Department, Atlanta Public Schools, dwhitlow@atlantapublicschools.us

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Explanation of the Terms and Categories Used in the Lesson Template

Task: This section gives the suggested number of days needed to teach the concepts addressed in a task, the task name, and the problem numbers of the task as listed in the Georgia Department of Education's Mathematics II Framework Teacher Edition (GaDOE TE).

In some cases new tasks or activities have been developed. These activities have been named by the writers.

Standard(s): Although each task addresses many Math II standards and uses mathematics learned in earlier grades, in this section, only the key standards addressed in the lesson are listed.

New Vocabulary: Vocabulary is listed here the *first* time it is used. It is strongly recommended that teachers, particularly those teaching Math Support, use interactive word walls. Vocabulary listed in this section should be included on the word walls and previewed in Math Support.

Mathematical concepts/skills: Major concepts addressed in the lesson are listed in this section whether they are Math II concepts or were addressed in earlier grades or courses.

Prior knowledge: Prior knowledge includes only those topics studied in previous grades or courses. It does not include Math II content taught in previous lessons.

Essential Question(s): Essential questions may be daily and/or unit questions.

Suggested materials: This is an attempt to list all materials that will be needed for the lesson, including consumable items, such as graph paper; and tools, such as graphing calculators and compasses. This list does not include those items that should always be present in a standards-based mathematics classroom such as markers, chart paper, and rulers.

Warm-up: A suggested warm-up is included with every lesson. Warm-ups should be brief and should focus student thinking on the concepts that are to be addressed in the lesson.

Opening: Openings should set the stage for the mathematics to be done during the work time. The amount of class time used for an opening will vary from day-to-day but this should not be the longest part of the lesson.

Worktime: The problem numbers have been listed and the work that students are to do during the worktime has been described. A student version of the day's activity follows the lesson template in every case. In order to address all of the standards in Math II, some of the problems in some of the original GaDOE tasks have been omitted and less time consuming activities have been substituted for those problems. In many instances, in the student versions of the tasks, the writing of the original tasks has been simplified. In order to preserve all vocabulary, content, and meaning it is important that teachers work the original tasks as well as the student versions included here.

Teachers are expected to both facilitate and provide some direct instruction, when necessary, during the work time. Suggestions related to student misconceptions, difficult concepts, and deeper meaning have been included in this section. However, the teacher notes in the GaDOE Math II Framework are exceptional. In most cases, there is no need to repeat the information provided there. Again, it is imperative that teachers work the tasks and read the teacher notes that are provided in GaDOE support materials.

Questioning is extremely important in every part of a standards-based lesson. We included suggestions for questions in some cases but did not focus on providing good questions as extensively as we would have liked. Developing good questions related to a specific lesson should be a focus of collaborative planning time.

Closing: The closing may be the most important part of the lesson. This is where the mathematics is formalized. Even when a lesson must be continued to the next day, teachers should stop, leaving enough time to "close", summarizing and formalizing what students have done to that point. As much as possible students should assist in presenting the mathematics learned in the lesson. The teacher notes are all important in determining what mathematics should be included in the closing.

Homework: In some cases, homework suggestions are provided. Teachers should use their resources, including the textbook, to assign homework that addresses the needs of their students.

Homework should be focused on the skills and concepts presented in class, relatively short (30 to 45 minutes), and include a balance of skills and thought-provoking problems.

Differentiated support/enrichment: On the topic of differentiation, it is critical to reiterate that many of the strategies used in a standards-based mathematics classroom promote differentiation. These strategies include

- the use of rich tasks with multiple points of entry and more than one path to a solution
- flexible grouping of students
- multiple representations of mathematical concepts
- writing in mathematics
- monitoring of progress through on-going informal and formative assessments
- and analysis of student work.

Check for understanding: A check for understanding is a short, focused assessment-a ticket out the door, for example. There are many good resources for these items, including the GaDOE culminating task at the end of each unit and the *Mathematics II End-of-Course Study Guide*. Both resources can be found on-line at www.georiastandards.org, along with other GaDOE materials related to the standards. Problem numbers from the GaDOE culminating task have been listed with the appropriate lessons in this document.

Resources/materials for Math Support: Again, in some cases, we have provided materials and/or suggestions for Math Support. This section should be personalized to your students, class, and/or school, based on your resources.

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Unit 1 Timeline

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Task 7: Geometric Connections	2 days

Task Notes

The importance of working the tasks used in these lessons cannot be overstated. In planning for the teaching of the Georgia Performance Standards in Mathematics, teachers should work the task, read the teacher notes provided in the Georgia Department of Education's Mathematics II Framework Teacher Edition, and *then* examine the lessons provided here.

Task 1: Henley's Chocolates

The big ideas presented in this task include:

- Determining a quadratic equation to model a real situation;
- Review of transformations of $f(x) = x^2$ introduced in Math I (vertical stretches and shrinks, reflections, and vertical shifts);
- Horizontal shifts of $f(x) = x^2$; and
- Solving quadratics equations by taking square roots.

A problem has been added to the beginning of the GaDOE task requiring students to "construct" a box, using centimeter grid paper that satisfies the specifications of Henley's Chocolates. All parts of the GaDOE TE task are included in these lesson plans with some revisions in wording, particularly in GaDOE problem #7. GaDOE Teacher Notes are still applicable for all problems.

Task 2: The Protein Bar Toss

The big ideas presented in this task include:

- Using appropriate graphing technology to relate the characteristics of the graph of a general quadratic function to a context modeled by the function;
- Factoring general quadratic polynomials by grouping; and
- Solving general quadratic equations by factoring.

Some problems from the original GaDOE task have been used as openings, assigned as homework, or condensed in the interest of time. Items have been renumbered but all problems of the original GaDOE task are addressed in these plans.

Task 3: The protein Bar Toss, Part 2

The big ideas presented in this task include:

- Finding the vertex of the graph of a general quadratic function, and

  (Note: The method used here for finding the vertex involves the y-intercept, the point on the parabola symmetric to the y-intercept, and the axis of symmetry. Students do *not* complete the square to find the vertex.)

- Converting between standard and vertex form of a quadratic function.

Problems from the original GaDOE task have been revised but all items are addressed and GaDOE teacher notes are still applicable. It is important that students finish and discuss all parts of the student task in these lesson plans. Problem 14 is particularly important in that it foreshadows the work to be done in the next task.

Task 4: Paula's Peaches Revisited

The big ideas presented in this task include:

- Solving quadratic inequalities graphically and algebraically,
- Solving general quadratic equations using vertex form and extraction of roots, and
- The introduction of quadratic equations with no real solutions.

Problems from the original GaDOE task have been revised and parts of the task are treated as whole class discussion. GaDOE problem 3 has been altered to include the technique of plotting points of equality on the number line and testing truth values for the corresponding inequality.

Introduction of the quadratic formula has been moved to the next task where it will be immediately applied.
GaDOE teacher notes are applicable.

Task 5: Just the right Border

The big ideas presented in this task include:

- Introduction and proof of the quadratic formula,
- Using the quadratic formula to solve equations with real coefficients,
- Using graphing technology to verifying solutions of quadratic equations by finding the *x*-intercepts of corresponding quadratic functions, and
- Analyzing the nature of the roots of a quadratic equation using the discriminant.

The introduction of the quadratic formula has been moved from the previous task to immediately follow the determination of the equation in problem 1 of this task. Students will compare the efficiency of using the "vertex" method to solve the equation with that of using the quadratic formula.

All parts of the original GaDOE task are addressed in these plans.

Task 6: Imagining a New Number

The big ideas presented in this task include:

- The need for complex numbers,
- Simplifying and operating with complex numbers,
- Graphing in the complex plane, and
- Finding complex solutions of quadratic equations.

The history of square roots of negative numbers has been greatly reduced from that of the original GaDOE task. Problems 1, 2, and 5 of the original task have been omitted. Other items have been used as warm-ups, openings, and homework in an effort to reduce the amount of time needed for students to effectively learn the material. The homework included at the end of Day 2 gives students an opportunity to solve quadratic equations with complex solutions. These problems should be completed by all students and thoroughly discussed in class.

Task 7: Geometric Connections

The big ideas presented in this task include:

- Exploring various ways of finding sums of arithmetic series,
- Exploring sequences of partial sums of arithmetic series
- Representing the n^{th} term of a sequence of partial sums of arithmetic series in closed form as a quadratic function
- Using sums of arithmetic series to solve real world problems

Problems 5 – 8 of the original GaDOE task have been omitted. All other items have been addressed and GaDOE teacher notes are applicable.

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Teacher's Curriculum Supplement

Mathematics II: Unit 1

Task 1: Henley's Chocolates

Mathematics II
Task 1: Henley's Chocolates **Day 1/3**
(GaDOE TE # 1 - #4)

MM2A3. Students will analyze quadratic functions in the forms $f(x) = ax^2 + bx + c$ and $f(x) = a(x - h)^2 + k$.
b. Graph quadratic functions as transformations of the function $f(x) = x^2$.
c. Investigate and explain characteristics of quadratic functions, including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, intervals of increase and decrease, and rates of change.

New vocabulary:

Mathematical concepts/skills:
- area,
- translating from verbal to algebraic expressions,
- proportional relationships versus linear relationships that are not directly proportional,
- writing quadratic functions that model a given situation, and
- domains of functions used to model a given situation

Prior knowledge:
- area,
- translating from verbal to algebraic expressions,
- proportional relationships versus linear relationships that are not directly proportional, and
- finding domains of functions used to model a given situation

Essential question(s): How can I use quadratic functions to model real situations and solve problems related to the given situation?

Suggested materials: centimeter graph paper, centimeter rulers, scissors, and tape

Warm-up: The problems below are designed to minimize issues students may have in simplifying the function determined in problem 4 of the task.

Post the following:

Write each of the following expressions as a single fraction in simplest form:

1. $\frac{1}{1.5}$

2. $\frac{2}{3.6}$

3. $\frac{\frac{1}{2}}{4}$

Opening: Discuss the warm-up. Answers should be $\frac{2}{3}$, $\frac{5}{9}$, and $\frac{1}{8}$ respectively.

Give each student a copy of the student task. Ask students to read the scenario (down to problem 1) silently, making notes in the margin of important information. Ask them to also note any vocabulary they do not fully understand.

Have a student read the scenario aloud and then ask other students to describe the situation, being sure to note important facts and to define any vocabulary that students do not understand.

Worktime: Allow students to work in pairs or groups. Give each group of students a sheet of centimeter grid paper, scissors, and tape. Have them complete problem 1.

After students have completed problem 1, have a whole-class discussion on the different size boxes that groups have constructed. Completing a table like the one below, during the discussion, may help all students gain the understanding needed to complete the task.

w (width of box base in centimeters)	l (length of box base in centimeters)	W (width of cardboard in centimeters)	L (length of cardboard in centimeters)

An 8 ½ by 11 sheet of grid paper is approximately 21.6 cm by 28 cm. Some possible dimensions for student boxes are shown below.

w (width of box base in centimeters)	l (length of box base in centimeters)	W (width of cardboard in centimeters)	L (length of cardboard in centimeters)
1	2.5	9	10.5
2	5	10	13
4	10	12	18
6	15	14	23
8	20	16	28

A discussion on proportionality may be useful here. (See teacher notes.) Good questions to ask include:

What relationship exists between l (the length of the box base) and w (the width of the box base)? *The relationship is linear and directly proportional. $l = 2.5w$. Students will probably first see the constant rate of change and determine that the relationship is linear. They should be guided to see that it is directly proportional as well. Geometrically, all rectangles representing bases of these boxes are similar, meaning that ratios of corresponding sides are equal.*

Does this same relationship exist between W and L? Why or why not? *The relationship between W and L is linear ($L = 2.5\,W – 12$) but not directly proportional. It is not necessary here to dwell on having students find the linear equation for this relationship but it is advantageous for them to see it. Geometrically, the rectangles created by the pieces of cardboard necessary to create the boxes are not similar. Adding a constant value to the dimensions of a figure, will not create a similar figure.*

Students should complete problems 2 – 5 of the task. Monitor work carefully to be sure that all students understand that the function A is to be written in terms of L (the length of the cardboard). Once they have written the function, encourage students to leave it in the form $y = a(x – h)^2$, since our purpose here is to examine horizontal shifts.

Closing: Allow students to present/discuss problems 2 - 5 of the task. Make sure all students have arrived at the correct function and domain for problems 4 and 5. It will be useful to have both the function and its domain posted on an Anchor chart for Day 3 of the task.

Homework: The objectives of the homework following the student task are:

- to help students review transformations studied in Math I (vertical stretches and shrinks, vertical shifts, and reflections), and
- to acquaint students with some of the language included in the next lesson.

Differentiated support/enrichment:

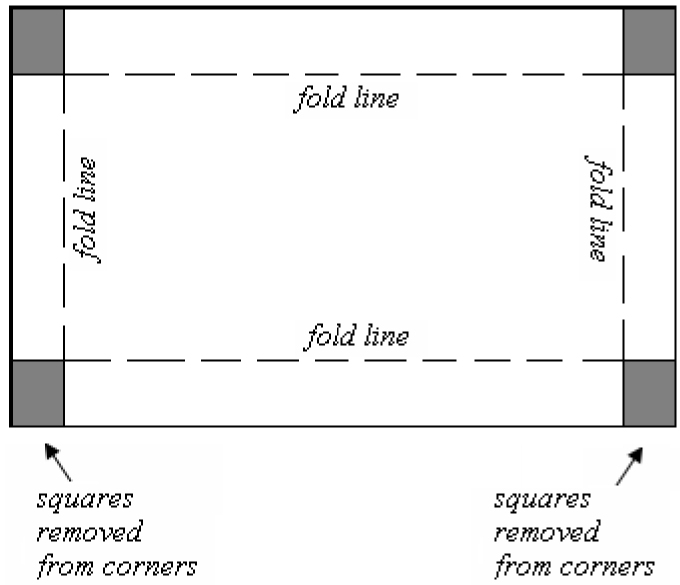
Check for Understanding:

Resources/materials for Math Support: Critical reading is important for all students, particularly those students who struggle in academic areas. This task gives an opportunity to focus on critical reading skills. You might want to begin by making sure that all students know what a truffle looks like. Ideas include bringing in a box of candy or showing a picture of a box of truffles. Allow students to guess how many pieces of candy are in the box and ask them how they arrived at their guess. Discuss the dimensions of a piece of candy. How big do they think each piece is? Students may also need help in distinguishing between the lengths and widths of the box bases and the lengths and widths of the pieces of cardboard required when building the box.

Skills preview may include simplifying complex fractions similar to those in the warm-up, finding the area of a rectangle, translating from verbal to algebraic expressions, graphing the function $f(x) = x^2$ and the transformations of $f(x) = x^2$ studied in Math I.

Mathematics II
Henley's Chocolates
Day 1 Student Task

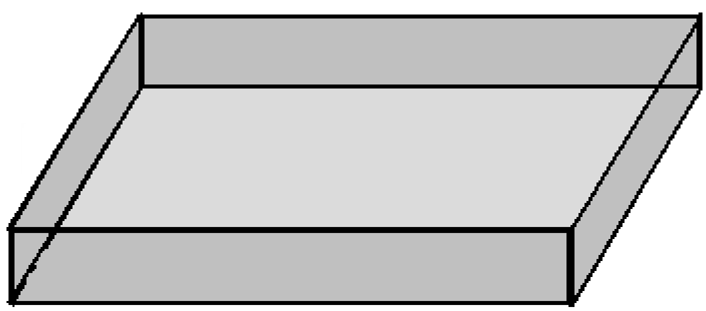
Henley's Chocolates is famous for its mini chocolate truffles, which are packaged in foil covered boxes. The base of each box is created by cutting squares with side measures of 4 centimeters from each corner of a rectangular piece of cardboard and folding the cardboard edges up to create a rectangular prism 4 centimeters deep. A matching lid is constructed in a similar manner, but, for this task, we focus on the base, which is illustrated in the diagrams below.



For the base of the truffle box, paper tape is used to join the cut edges at each corner. Then the inside and outside of the truffle box base are covered in foil.

Henley's Chocolates sells to a variety of retailers and creates specific box sizes in response to requests from particular clients. However, Henley's Chocolates requires that their truffle boxes always be 4 cm deep and that, in order to preserve the distinctive shape associated with Henley's Chocolates, the bottom of each truffle box be a rectangle that is two and one-half times as long as it is wide.

1. Use your centimeter grid paper to create the base of a box that will satisfy the requirements of Henley's chocolates. What are the dimensions of your box? Explain how you know the requirements are satisfied.



base of the truffle box

2. Henley's Chocolates restricts box sizes to those which will hold plastic trays for a whole number of mini truffles. A box needs to be at least 2 centimeters wide to hold one row of mini truffles. Let L denote the length of a piece of cardboard from which a truffle box is made. What value of L corresponds to a finished box base for which the bottom is a rectangle that is 2 centimeters wide?

3. Henley's Chocolates has a maximum size box of mini truffles that it will produce for retail sale. For this box, the bottom of the truffle box base is a rectangle that is 50 centimeters long. What are the dimensions of the piece of cardboard from which this size truffle box base is made?

4. Since the mini truffle boxes are 4 centimeters deep, each box holds two layers of mini truffles. Thus, the number of truffles that can be packaged in a box depends on the number of truffles that can be in one layer, and, hence, on the area of the bottom of the box. Let the function A denote the area, in square centimeters, of the rectangular bottom of a truffle box base. Write a formula for A in terms of the length $\boldsymbol{L}$, in centimeters, of the piece of cardboard from which the truffle box base is constructed.

5. Although Henley's Chocolates restricts truffle box sizes to those that fit the plastic trays for a whole number of mini truffles, the engineers responsible for box design find it simpler to study the function $A(L)$ on the domain of all real number values of L in the interval from the minimum value of L found in item 1 to the maximum value of L found in item 2. State this interval of L values as studied by the engineers at Henley's Chocolates.

Mathematics II
Henley's Chocolates
Day 1 Homework

Graph each of the functions below, without using graphing technology, on the same axes with the graph of the function $f(x) = x^2$. Use a new set of axes for each problem. For each function listed, describe the transformation(s) of the graph of f that results in the graph of the given function.

1. $f(x) = x^2; g(x) = 4x^2$

2. $f(x) = x^2; h(x) = -0.5x^2$

3. $f(x) = x^2; k(x) = x^2 - 4$

Mathematics II
Task 1: Henley's Chocolates **Day 2/3**
(GaDOE TE #5 - #8)

MM2A3. Students will analyze quadratic functions in the forms $f(x) = ax^2 + bx + c$ and $f(x) = a(x - h)^2 + k$.
b. Graph quadratic functions as transformations of the function $f(x) = x^2$.

New vocabulary: rigid transformation, horizontal shift

Mathematical concepts/skills:

- transformations of the function $f(x) = x^2$,
- including reflections,
- vertical stretches and shrinks,
- vertical and horizontal shifts; and
- the graphing features of the TI-83/84 family of calculators

Prior knowledge:

- transformations of the function $f(x) = x^2$,
- including reflections, and
- vertical stretches and shrinks, and vertical shifts

Essential question(s): How does the graph of a function in the form $h(x) = (x - h)^2$ compare to the graph of $f(x) = x^2$?

Suggested materials: graphing calculators, graph paper

Warm-up: Give each student a graphing calculator and ask them to work with a partner to check the graphs and descriptions of transformations they completed for homework. If students have had little or no experience with a graphing utility, you may have them compare homework without the aid of technology.

Opening: Discuss the homework making sure that students are using appropriate vocabulary to describe transformations.

If students have little or no experience with a graphing utility, this is a good time to investigate the graphing features of the calculator. The movies in sections B, H, J1 and J2 of the following website may be useful in helping both teachers and students better utilize any of the calculators in the TI-82, 83, or 84 families.

http://movies.atomiclearning.com/k12/ti_84/

After students have spent some time familiarizing themselves with the graphing features of the calculator, remind them of the function $A(L) = .4(L - 8)^2$ developed in the previous lesson. Ask if they think this function is related to the basic function $f(x) = x^2$. If so, how? If not, why not? Tell them that the task today is designed to help them answer this question.

Worktime: Students should work in pairs to complete problems 6 - 8 of the student task. They may have questions about the phrase "*rigid transformation*" in problems 6 and 7.

A transformation is rigid if it preserves distances. Translations and reflections are examples of rigid transformations. Stretches and shrinks (dilations) are not examples of rigid transformations.

Closing: The closing discussion should focus on problem 8. Students might share one set of graphs from problem 6 and state the transformation that occurred in these items. Then repeat this process for problem 7. (Note: You may need to remind students that $f(x)$ is equal to y, thus $f(x)$ and y, are interchangeable.) Each item in problem 8 should be discussed and students should arrive at the understanding contained in the following statement:

The graph of any function written in the form $y = (x - h)^2$ is obtained by shifting the graph of $f(x) = x^2$ by $|h|$ units to the right when h is positive, and by $|h|$ units to the left when h is negative.

(See GaDOE teacher notes, problem #7)

Homework: The homework given below is problem 8 in the GaDOE TE. For homework, students will *predict* how they think the graphs will be related. They will check their predictions using graphing technology as the warm-up for the next lesson.

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: Preview may include computation with positive and negative numbers and the graphing features of the TI calculator.

The concept contained in the generalization of horizontal shifts is a difficult one. Focusing on tables of values is often helpful for students who struggle with these concepts.

Mathematics II
Henley's Chocolates
Day 2 Student Task

6. **Use technology to graph** each of the following functions on the same axes with the graph of the function f defined by $f(x) = x^2$. Use a new set of axes for each function listed below, but repeat the graph of f each time. For each function listed, describe a rigid transformation of the graph of f that results in the graph of the given function.

   a. $y = (x-3)^2$

   b. $y = (x-6)^2$

   c. $y = (x-8)^2$

   Make a conjecture about the graph of $y = (x-h)^2$, where h is any real number.

7. **Use technology to graph** each of the following functions on the same axes with the graph of the function f defined by $f(x) = x^2$. Use a new set of axes for each function listed below, but repeat the graph of f each time. For each function listed, describe a rigid transformation of the graph of f that results in the graph of the given function.

   a. $y = (x+2)^2$

   b. $y = (x+5)^2$

   c. $y = (x+9)^2$

8. Look again at the exercises in items 6 and 7.

   a. We can view the exercises in item 6 as taking a function, in this case the function, $f(x) = x^2$, and replacing the "x" in the formula with "$x - h$". What is "$x - h$" in item 6a? What is "$x - h$" in item 6b?

   b. We can view the exercises in item 7 as replacing the "x" in the formula with "$x + h$", but we can also view these exercises as replacing the "x" in the formula with "$x - h$". What is "$x + h$" in item 7a? How could we view this expression as "$x - h$"? What is "$x + h$" in item 7b? How could we view this expression as "$x - h$"?

   c. Make a conjecture about the graph of the function $y = (x - h)^2$, where h is any real number. Does your conjecture describe the transformations that occur in both items 6 and 7? If so, explain how it works. If not, you may need to adjust your conjecture.

   d. What do you think will happen if we replace the "x" with "$x - h$" in the formulas for other functions in our basic family of functions? For example, how do you think the graphs of $f(x) = x^3$ and $h(x) = (x - 4)^3$ will compare? How about the graphs of $f(x) = |x|$ and $h(x) = |x + 5|$?

Mathematics II
Henley's Chocolates
Day 2 Homework

For each pair of functions below, predict how you think the graphs will be related. During your next class period, you will **use technology to graph** the two functions on the same axes and check your prediction.

a. $y = x^2$ and $y = 3x^2$

b. $y = 3x^2$ and $y = 3(x-4)^2$

c. $y = \frac{1}{2}x^2$ and $y = -\frac{1}{2}x^2 + 5$

d. $y = -0.75x^2$ and $y = 0.75(x+6)^2$

e. $y = 2x^2$ and $y = -2(x-5)^2 + 7$

Mathematics II
Task 1: Henley's Chocolates **Day 3/3**
(GaDOE TE #9 - #12)

MM2A3. Students will analyze quadratic functions in the forms $f(x) = ax^2 + bx + c$ and $f(x) = a(x - h)^2 + k$.

b. Graph quadratic functions as transformations of the function $f(x) = x^2$.
c. Investigate and explain characteristics of quadratic functions, including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, intervals of increase and decrease, and rates of change.

MM2A4. Students will solve quadratic equations and inequalities in one variable.

a. Solve equations graphically using appropriate technology.
b. Find real and complex solutions of equations by factoring, taking square roots, and applying the quadratic formula.

New vocabulary:

Mathematical concepts/skills:

- transformations of the function $f(x) = x^2$, including vertical stretches and shrinks, and horizontal shifts;
- domain and range;
- solving quadratic equations by extraction of roots; and
- finding solutions to problems in context

Prior knowledge:

- transformations of the function $f(x) = x^2$, including reflections, vertical stretches and shrinks, and vertical shifts;
- domain and range; and
- solving simple quadratic equations by extraction of roots

Essential question(s): How can I use quadratic functions to model real situations and to solve problems related to the given situation?

Suggested materials: graphing calculators, graph paper

Warm-up: Give each student a graphing calculator and ask them to work with a partner to check the predictions they made in the homework assignment by graphing the assigned functions.

Opening: Discuss the homework. At this point, all students should be able to describe transformations of the function $f(x) = x^2$, given an equation in the form $y = a(x - h)^2 + k$.

Transformations include vertical stretches and shrinks, reflections, and vertical and horizontal shifts.

Remind students of the function $A(L) = .4(L-8)^2$ and its domain $13 \le L \le 58$, derived on Day 1 of the task . Tell them they will use this function to complete the task and to help the engineers at Henley's Chocolates answer some important questions.

Worktime: Students should work in pairs to complete problems 9 - 12 of the student task. Make sure students give both the positive and negative solutions to the equations that arise in problem 11 but distinguish between the two solutions when solving the real world problems.

Closing: Allow students to present their work. (See GaDOE teacher notes.)

Homework:

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: Students may benefit from a scenario similar to the one provided in Henley's Chocolates in which they are asked similar questions.
Skills preview may include solving simple equations by extraction of roots.

Mathematics II
Henley's Chocolates
Day 3 Student Task

Now we return to the function studied by the engineers at Henley's Chocolates.

9. Let $g(x)$ be the function with the same formula as the formula for function $A(L)$ but with domain all real numbers, rather than the restricted domain of the function A.

   a. Describe the transformations of the function $f(x) = x^2$, that will produce the graph of the function g.

   b. Use **technology to graph** f and g on the same axes to check that the graphs match your description of the described transformations.

10. Describe the graph of the function A, used by the engineers at Henley's Chocolates, in words and make a hand drawn sketch. Remember that you found the domain of the function in problem 5. What is the range of the function A?

11. The engineers at Henley's Chocolates responsible for box design have decided on two new box sizes that they will introduce for the next winter holiday season.

    a. The area of the bottom of the larger of the new boxes will be 640 square centimeters. Use the function A to write and solve an equation to find the length L of the cardboard needed to make this new box.

    b. The area of the bottom of the smaller of the new boxes will be 40 square centimeters. Use the function A to write and solve an equation to find the length L of the cardboard needed to make this new box.

12. How many mini-truffles do you think the engineers plan to put in each of the new boxes?

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Teacher's Curriculum Supplement

Mathematics II: Unit 1

Task 2: The Protein Bar Toss

Mathematics II
Task 2: The Protein Bar Toss **Day 1/3**
(GaDOE TE # 1 – 5)

MM2A3. Students will analyze quadratic functions in the forms $f(x) = ax^2 + bx + c$ and $f(x) = a(x - h)^2 + k$.

c. Investigate and explain characteristics of quadratic functions, including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, intervals of increase and decrease, and rates of change.

MM2A4. Students will solve quadratic equations and inequalities in one variable.

a. Solve equations graphically using appropriate technology.
b. Find real and complex solutions of equations by factoring, taking square roots, and applying the quadratic formula.

New vocabulary:

Mathematical concepts/skills:

- relating the characteristics of a general quadratic function (domain, range, x-intercept(s) and y- intercept) to a given context utilizing appropriate graphing technology;
- solving quadratic equations graphically and algebraically (by factoring); and
- zero product property

Prior knowledge:

- relating the characteristics of a function to a given context utilizing appropriate graphing technology;
- solving simple quadratic equations graphically and algebraically (by factoring when $a = 1$); and
- zero product property

Essential question(s): How can I use quadratic functions to model real situations and to solve problems related to the given situation?

Suggested materials: graphing utility, graph paper

Warm-up: Post the following.

In Math I, you investigated a formula for the distance fallen by an object dropped from a high place. The formula was similar to the function given below. Work problems 1 – 3 in preparation for today's lesson.

If a ball is dropped from a high place, such as the Tower of Pisa in Italy, then there is a formula for calculating the distance the ball has fallen. If y, measured in meters, is the distance the ball has fallen and t, measured in seconds, is the time since the ball was dropped, then y is a function of t, and the relationship can be approximated by the formula $y = d(t)$ and $y = 4.9\,t^2$. Here we name the function d because the outputs are distances.

t	0	1	2	3	4	5	6	…
$y = d(t) = 4.9t^2$								…

1. Fill in the missing values in the table above.
2. Suppose the ball is dropped from a building at least 100 meters high. Measuring from the top of the building, draw a picture indicating the position of the ball at times indicated in your table of values.
3. Draw a graph of y versus t for this situation.

Opening: Discuss the warm-up. Compare the picture and the graph and discuss the differences in the representations. Notes from the Math I TE are included below.

Both problems 2 and 3 give a type of pictorial representation of how the distance that the ball has fallen varies with time. In the picture in problem 2, the position of the ball is shown relative to a representation of the building, so the position of the ball gets lower as time passes. In the graph, the y-value represents the distance fallen so the y-values get larger as time passes. In the picture, the varying positions of the ball are shown on the same vertical line representing the actual path of the object. On the graph, there are two coordinates for each point. The first coordinate represents the time since the ball was dropped from the top of the building, so no two points are on the same vertical line.

After discussing the warm-up, give each student a copy of the student task. Ask students to read the scenario silently, making notes in the margin of important information.

Have a student read the scenario aloud and then ask other students to describe the situation. After a brief discussion, you may want to allow students to reenact the toss. It is important to note that the bar "went straight up". Much like the object dropped from the top of a building, the path of the bar is not going to look like the graph of the quadratic function that shows its height from the ground at a given time t. (Note: If you really want to generate some conversation among your students, there is actually a brand of protein bar called the Cliff Bar.)

After the reenactment, write the functions representing the object dropped from the building and the protein bar toss on the board. Ask students to compare the two situations. How are they alike? How are they different? (See GaDOE teacher notes, problem #1.)

Worktime: Students should work problems 1 – 4 of the student task. Notice that students are to write but not solve the equation needed to answer the question posed in problem 4. This problem sets the stage for factoring general quadratic equations.

Closing: Allow students to share their work. (See GaDOE Teacher Notes, problems 2 – 5.) Creating an Anchor chart with the function $h(t)$, its graph, and the equation written for problem 4 will help remind students of the work done in this lesson when they begin Day 3 of the task.

Homework: Problems 6 and 7 of the GaDOE task are assigned for homework. Before assigning the problems, you may ask students to multiply the binomials in problem 6a and represent the product using an area model. These problems should be review for all students who completed Math I.

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: Students should preview situations that can be modeled by general quadratic functions. Questions similar to those included in this task should be asked.
Skills preview may include problems similar to GaDOE problems 6 and 7.

Mathematics II
The Protein Bar Toss
Day 1 Student Task

Blake and Zoe were hiking in a wilderness area. They came up to a scenic view at the edge of a cliff. As they stood enjoying the view, Zoe asked Blake if he still had some protein bars left, and, if so, could she have one. Blake said, "Here's one; catch!" As he said this, he pulled a protein bar out of his backpack and threw it up to toss it to Zoe. But the bar slipped out of his hand sooner than he intended, and the bar went straight up in the air with his arm out over the edge of the cliff. The protein bar left Blake's hand moving straight up at a speed of 24 feet per second. If we let t represent the number of seconds since the protein bar left the Blake's hand and let $h(t)$ denote the height of the bar, in feet above the ground at the base of the cliff, then, assuming that we can ignore the air resistance, we have the following formula expressing $h(t)$ as a function of *t,*

$$h(t) = -16t^2 + 24t + 160.$$

In this formula, the coefficient on the t^2-term is due to the effect of gravity and the coefficient on the t-term is due to the initial speed of the protein bar caused by Blake's throw. In this task, you will explore, among many things, the source of the constant term.

1. **Use technology to graph the equation** $y = -16t^2 + 24t + 160$. Find a viewing window that includes the part of this graph that corresponds to the situation with Blake and his toss of the protein bar. What viewing window did you select?
   Copy the graph onto your graph paper.

2. What was the height of the protein bar, measured from the ground at the base of the cliff, at the instant that it left Blake's hand? What special point on the graph is associated with this information?

3. If Blake wants to reach out and catch the protein bar on its way down, how many seconds does he have to process what happened and position himself to catch it? Justify your answer graphically and algebraically.

4. If Blake does not catch the falling protein bar, how long will it take for the protein bar to hit the ground below the cliff? Justify your answer graphically. Then write a quadratic equation that you would need to solve to justify the answer algebraically.

The equation that you have written in problem 4 can be solved by factoring, but it requires factoring a quadratic polynomial where the coefficient of the x^2-term is not 1. The goal of our next lesson is to learn about factoring this type of polynomial. The homework that you complete tonight will examine products that lead to such quadratic polynomials.

Mathematics II
The Protein Bar Toss
Day 1 Homework

1. For each of the following, perform the indicated multiplication and use a rectangular model to show a geometric interpretation of the product as area for positive values of x.

   a. $(2x+3)(3x+4)$
   b. $(x+2)(4x+11)$
   c. $(2x+1)(5x+4)$

2. For each of the following, perform the indicated multiplication.

   a. $(2x-3)(9x+2)$
   b. $(3x-1)(x-4)$
   c. $(4x-7)(2x+9)$

Mathematics II
Task 2: The Protein Bar Toss **Day 2/3**
(GaDOE TE # 8 - 11)

MM2A4. Students will solve quadratic equations and inequalities in one variable.
d. Find real and complex solutions of equations by factoring, taking square roots, and applying the quadratic formula.

New vocabulary: complete factorization of a polynomial over the set of integers

Mathematical concepts/skills:
- multiplying binomials to obtain a quadratic polynomial, and
- factoring general quadratic polynomials over the set of integers

Prior knowledge:
- multiplying binomials to obtain a quadratic trinomial, and
- factoring quadratic polynomials in the form $ax^2 + bx + c$, where $a = 1$

Essential question(s): How do I know that a quadratic polynomial is factored completely over the set of integers?

Suggested materials:

Warm-up: Allow students to compare homework with a partner. When finished ask them to do the following:

Use the distributive property to write each of the following polynomials as the product of the greatest common factor of each term and the remaining polynomial expression.

1. $5x^3 – 15x^2$
2. $3y^4 – 6y^3 + 18y^2$
3. $2xy^2 + 6xy – 8x$

Opening: Discuss any homework problems about which students still have questions. Include both the products of the binomials and the area models representing those products in the discussion.

The remaining objective of this opening is to remind students of two procedures addressed in Math I:
- factoring out the greatest factor common to all terms in a polynomial, and
- factoring by grouping.

Being comfortable with these procedures will allow students to concentrate on the methods used to factor general quadratic polynomials presented in this lesson.

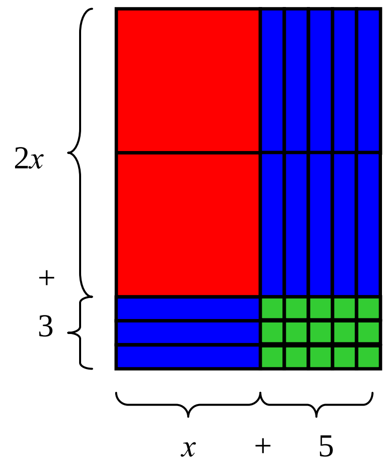
Discuss problems 1 – 3 above, making sure that all students understand how to factor out the greatest common factor of the terms in a polynomial.

Ask students to factor the polynomial below by grouping. (Students may need to see more than one example of factoring by grouping.)

$2x^2 + 3x + 10x + 15$

Using both the algebraic representation and the area model below should help students better understand the procedure.

$2x^2 + 3x + 10x + 15$
$(2x^2 + 3x) + (10x + 15)$
$x(2x + 3) + 5(2x + 3)$
$(2x + 3)(x + 5)$



Worktime: Put students in groups of 3 or 4. Give each student a copy of the student task. Write one of the quadratic polynomials below on an index card. Allow each group of students to draw one card. The polynomial they draw will be used to complete problem 5 of the student task. Have each group of students create an Anchor chart illustrating the procedure for factoring outlined in problem 5. Student charts should include the table of integer pairs whose product is *ac*, the polynomial rewritten by replacing *bx* with *mx* + *nx,* factoring by grouping (hopefully illustrated by the area model shared in the opening), the factored polynomial, and a check of the factoring by multiplying the determined factors.

After groups have had time to create their charts, give students time to work the problems not assigned to their group. These problems are found in problems 6 of the student task. Discussion is encouraged but each student should have these problems worked on their own paper.

After students have had time to finish group work and individual problems, have groups share their charts as a means of checking work, addressing misconceptions, and reinforcing the procedure.

a. $6x^2 + 7x - 20$
b. $2x^2 + 3x - 54$
c. $4w^2 - 11w + 6$
d. $3t^2 - 13t - 10$
e. $8x^2 + 5x - 3$
f. $18z^2 + 17z + 4$
g. $6p^2 - 49p + 8$

Students should finish problems 7 and 8, if time allows. If time does not allow, problem 8 may be completed for homework but should be discussed thoroughly in the next class period.

Closing: Discuss the question asked in problem 6: *Is it necessary to always list all the integer pairs whose product is ac?* Students should begin to see that there are shortcuts to this process. Also discuss problems 7 and 8. Make sure students understand that it is customary to factor out the greatest common factor of a polynomial. (See GaDOE teacher notes.)

Homework: Provide students with more practice in factoring general quadratic polynomials.

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: Students should preview factoring out the greatest common factor, factoring by grouping, and factoring quadratic trinomials in the form $ax^2 + bx + c$, where $a = 1$.

Terminology including *factorization* and *complete factorization over the set of integers* should be emphasized.

Extra practice in factoring general quadratic polynomials should be provided. One way to get students practicing factoring *and* up and moving by using a strategy called “Around the Room”. Simply choose 5 – 10 problems. On chart paper, write the *answer* (in this case the factored form) to one problem at the top of the paper and write a different problem on the bottom of the sheet. Different students start at different places with the problem at the bottom of their starting sheet. They move around the room looking for the answer to their problem at the top of another sheet. When they find it, they work the problem at the bottom of that sheet. Students should end up at the same place in the room where they started.

Mathematics II
The Protein Bar Toss
Day 2 Student Task

5. Factor the quadratic polynomial assigned by your teacher using the following steps.
   a. Think of the polynomial as fitting the form $ax^2 + bx + c$.

      What is *a?* ____ What is *c*? ____ What is the product *ac*? ____

   b. List all possible pairs of integers such that their product is equal to the number *ac*. It may be helpful to organize your list in a table. Make sure that your integers are chosen so that their product has the same sign, positive or negative, as the number *ac* from above, and make sure that you list all of the possibilities.

   c. What is *b* in the quadratic polynomial given? _____ Add the integers from each pair listed in part b. Which pair adds to the value of *b* from your quadratic polynomial? We'll refer to the integers from this pair as *m* and *n*.

   d. Rewrite the polynomial replacing *bx* with $mx + nx$. [*Note either m or n could be negative; the expression indicates to add the terms mx and nx including the correct sign.*]

   e. Factor the polynomial from part d by grouping.

   f. Check your answer by performing the indicated multiplication in your factored polynomial. Did you get the original polynomial back?

6. Use the method outlined in the steps of item 5 to factor each of the following quadratic polynomials, except the polynomial assigned to your group. Is it necessary to always list all of the integer pairs whose product is *ac*? Explain your answer.

   a. $6x^2 + 7x - 20$
   b. $2x^2 + 3x - 54$
   c. $4w^2 - 11w + 6$
   d. $3t^2 - 13t - 10$
   e. $8x^2 + 5x - 3$
   f. $18z^2 + 17z + 4$
   g. $6p^2 - 49p + 8$

7. If you are reading this, then you should have factored all of the quadratic polynomials listed in item 6 in the form $(Ax+B)(Cx+D)$, where the *A, B, C,* and *D* are all integers.

   a. Compare your answers with other students, or other groups of students. Did everyone in the class write their answers in the same way? Explain how answers can look different but be equivalent.

   b. Factor $24q^2-4q-8$ completely.

   c. Show that $24q^2-4q-8$ can be factored in the form $(Ax+B)(Cx+D)$, where the *A, B, C,* and *D* are all integers, using three different pairs of factors.

   d. How should answers to quadratic factoring questions be expressed so that everyone who works the problem correctly lists the same factors, just maybe not in the same order?

8. If a quadratic polynomial can be factored in the form $(Ax+B)(Cx+D)$, where the *A, B, C,* and *D* are all integers, the method you have been using will lead to the answer, specifically called the correct ***factorization***. As you continue your study of mathematics, you will learn ways to factor quadratic polynomials using numbers other than integers. For right now, however, we are interested in factors that use integer coefficients. Show that each of the quadratic polynomials below cannot be factored in the form $(Ax+B)(Cx+D)$, where the *A, B, C,* and *D* are all integers.

   a. $4z^2+z-6$
   b. t^2+2t+8
   c. $3x^2+15x-12$

Mathematics II
Task 2: The Protein Bar Toss **Day 3/3**
(GaDOE TE # 12 - 15)

MM2A3. Students will analyze quadratic functions in the forms $f(x) = ax^2 + bx + c$ and $f(x) = a(x - h)^2 + k$.
c. Investigate and explain characteristics of quadratic functions, including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, intervals of increase and decrease, and rates of change.

MM2A4. Students will solve quadratic equations and inequalities in one variable.
c. Solve equations graphically using appropriate technology.
d. Find real and complex solutions of equations by factoring, taking square roots, and applying the quadratic formula.

New vocabulary:

Mathematical concepts/skills:
- zero product property;
- solving general quadratic equations graphically and algebraically (by factoring); and
- relating the characteristics of a general quadratic function to a given context

Prior knowledge:
- zero product property; solving simple quadratic equations in the form
- $ax^2 + bx + c = 0$, where $a = 1$ graphically and algebraically (by factoring); and
- relating the characteristics of a quadratic function in the form $f(x) = ax^2 + bx + c$, where $a = 1$ to a given context

Essential question(s): How can I use quadratic functions to model real situations and to solve problems related to the given situation?

Suggested materials: graphing utility, graph paper

Warm-up: Ask students to compare homework and be prepared to ask questions related to those problems they still do not understand.

Post the following:

Solve the quadratic equations below by factoring.
1. $x^2 - x - 20 = 0$
2. $4x^2 + 12x = 84 - 28x$

Opening: This lesson focuses on two big ideas:
- solving general quadratic equations by factoring, and
- examining the constant term of a function in the form $f(x) = ax^2 + bx + c$, in the context of a given situation.

Begin the opening by answering any questions related to the homework on factoring general quadratic polynomials. Once these questions have been answered, allow students to discuss how they found solutions to the two quadratic equations above. Ask them how they know they have the correct solutions. Students should have learned to solve equations that can be written in the form $x^2 + bx + c = 0$ in Math I but may need review.

Worktime: Students should work problems 9 – 12 of the student task. The polynomials in problem 9 are the same polynomials that were factored in problems 6 and 7 of the previous lesson. Students should use the factorizations they have already obtained and the zero product property to solve these equations. It may not be necessary for students to complete all 7 of these problems in class. Once students have grasped the concepts involved, the remainder of these problems might be assigned as homework.

After students have demonstrated that they understand the concepts involved in problem 9, review the scenario of the Protein Bar Toss by referring to the Anchor chart created on Day 1 of the task. Remind them of the equation they created in problem 4. The question asked, and the equation written to answer the question, are as follows:

> *If Blake does not catch the falling protein bar, how long will it take for the protein bar to hit the ground below the cliff? Justify your answer graphically. Then write a quadratic equation that you would need to solve to justify the answer algebraically.*

$$-16t^2 + 24t + 160 = 0$$

Allow students time to finish problems 10 – 12. The focus of problems 11 and 12 is the role of the constant in the function $f(x) = ax^2 + bx + c$. Problem 12 asks the same questions as 11 with a different value for c. This problem could be finished for homework, if necessary.

Closing: Allow students to discuss their work. (See GaDOE Teacher Notes, problems 12 - 15.)

Homework:

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: Students should preview different situations that can be modeled by general quadratic functions. Questions similar to those included in this task should be asked.

Skills preview may include solving quadratic equations that can be written in the form $x^2 + bx + c = 0$. Be sure to include equations in which the polynomials contain a greatest common factor and which need some manipulation to be written in the form $x^2 + bx + c = 0$.

Mathematics II
The Protein Bar Toss
Day 3 Student Task

9. Recall that you solved quadratic equations of the form $ax^2 + bx + c = 0$, with a = 1, in Mathematics I. The method required factoring the quadratic polynomial and using the Zero Factor Property. The same method still applies when $a \neq 1$, its just that the factoring is more involved, as you saw in the previous lesson. You may use your factorizations from problems 6 and 7 to help you solve the quadratic equations below.

   a. $2x^2 + 3x - 54 = 0$
   b. $4w^2 + 6 = 11w$
   c. $3t^2 - 13t = 10$
   d. $2x(4x + 3) = 3 + x$
   e. $18z^2 + 21z = 4(z - 1)$
   f. $8 - 13p = 6p(6 - p)$
   g. $24q^2 = 4q + 8$

10. Now we return to our goal of answering the question asked in problem 4 of the task. Remember that the function used to represent the height of the protein bar from the ground at time t was

$$h(t) = -16t^2 + 24t + 160$$

where height was measured in feet and time was measured in seconds. You were asked to answer the following question:

If Blake does not catch the falling protein bar, how long will it take for the protein bar to hit the ground below the cliff?

You found the answer by examining the graph of the function and were asked to write an equation that would justify your answer algebraically. What equation did you write? Solve the equation. Does the solution of your equation support what you found graphically? Explain.

11. Suppose the cliff had been 56 feet higher. Answer the following questions for this higher cliff.

   a. What was the height of the protein bar, measured from the ground at the base of the cliff, at the instant that it left Blake's hand? What special point on the graph is associated with this information?

b. What is the formula for the height function in this situation?

c. If Blake wants to reach out and catch the protein bar on its way down, how many seconds does he have to process what has happened and position himself to catch it? Justify your answer algebraically.

d. If Blake does not catch the falling protein bar, how long it take for the protein bar to hit the ground below the cliff? Justify your answer algebraically.

12. Suppose the cliff had been 88 feet lower. Answer the following questions for this lower cliff.

a. What was the height of the protein bar, measured from the ground at the base of the cliff, at the instant that it left Blake's hand? What special point on the graph is associated with this information?

b. What is the formula for the height function in this situation?

c. If Blake wants to reach out and catch the protein bar on its way down, how many seconds does he have to process what has happened and position himself to catch it? Justify your answer algebraically.

d. If Blake does not catch the falling protein bar, how long it take for the protein bar to hit the ground below the cliff? Justify your answer algebraically.

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Teacher's Curriculum Supplement

Mathematics II: Unit 1

Task 3: The Protein Bar Toss, Part 2

Mathematics II
Task 3: The Protein Bar Toss, Part 2 **Day 1/2**
(GaDOE TE #1 - 7)

MM2A3. Students will analyze quadratic functions in the forms $f(x) = ax^2 + bx + c$ and $f(x) = a(x - h)^2 + k$.
a. Convert between standard and vertex form.
c. Investigate and explain characteristics of quadratic functions, including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, intervals of increase and decrease, and rates of change.

MM2A4. Students will solve quadratic equations and inequalities in one variable.
a. Solve equations graphically using appropriate technology.
b. Find real and complex solutions of equations by factoring, taking square roots, and applying the quadratic formula.

New vocabulary: parabola, vertex of a parabola

Mathematical concepts/skills:
- evaluating functions for given values of the domain;
- examining characteristics of parabolas, including *y*-intercepts, and axes of symmetry;
- using technology to estimate vertices of quadratic functions;
- solving general quadratic functions by factoring;
- finding the vertex of a quadratic function; and
- using graphical and algebraic methods to answer questions related to a given context modeled by a quadratic function

Prior knowledge:
- evaluating functions for given values of the domain;
- finding *x*- and *y*- intercepts; writing equations of vertical lines; and
- graphing transformations of $f(x) = x^2$, including vertical stretches and shrinks, and vertical shifts

Essential question(s): How can I find the vertex of any quadratic function?

Suggested materials: graphing utility, graph paper

Warm-up: Post the following:

You began your study of quadratic functions in Math I with the basic function $f(x) = x^2$ and some of its transformations. In Math II, you have investigated more general quadratic functions like those considered in Henley's Chocolates and the Protein Bar Toss. Every quadratic function has a formula that can be put in the form $y = ax^2 + bx + c$ where $a \neq 0$. If we consider these functions with domains that are unrestricted by some real context, in other words, with domain all real numbers, the shapes of their graphs are very similar. The *shape is called a parabola. List at least three characteristics common to the parabolas you have seen so far.*

Opening: Discuss characteristics of parabolas listed by students in the warm-up. (See teacher notes for problem #1 of the GaDOE task.)

After students seem comfortable with the general characteristics of a parabola, have a whole-class discussion of problems 1, 2a, and 2b of the student task. Remind students of the three functions (listed below) that were used in the Protein Bar Toss (Part 1).

$$h_1(t) = -16t^2 + 24t + 160$$
$$h_2(t) = -16t^2 + 24t + 216$$
$$h_3(t) = -16t^2 + 24t + 72$$

Worktime: Students should begin working the task at problem 2c. It is important for them to estimate the vertices of the graphs of h_1, h_2, and h_3 using a graphing utility. This sets up the need to explore algebraic methods for finding the exact values of the vertices. Note that students are led, in problem 5, to find the vertex of a parabola using the y-intercept, the point on the parabola symmetric to the y-intercept, and the axis of symmetry. They do *not* complete the square to find the axis of symmetry.

Closing: In problem 7b, students find that the x-coordinate of the vertex of the graph of any quadratic function can be found using the formula $x = \frac{-b}{2a}$ and that the y-coordinate of the vertex is always $f\left(\frac{-b}{2a}\right)$. Problem 8 reinforces this very important concept. A thorough discussion of both of these problems is extremely important. Once students have discovered these formulas, they should commit them to memory as they have in the past.

Homework: The homework assigned below includes problems 6b and 8 of the original GaDOE task. The first problem reinforces the method used in class to find the vertex. The second problem reinforces the formula for finding the x-coordinate of the vertex. The third problem requires students to find the exact coordinates of the vertices of h_1, h_2, and h_3 which were previously estimated from the graphs.

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: Preview should include evaluating functions for specific values of x, exploring x- and y- intercepts both in relation to graphs of quadratic functions and in context, determining axes of symmetry for parabolas and writing equations of axes of symmetry. Students may also need extra work on solving general quadratic equations by factoring.

Mathematics II
The Protein Bar Toss, Part 2
Day 1 Student Task

1. In the first part of the Protein Bar task, you found how long it took for the bar to go up and come back down to the starting height. However, there is a question you did not consider: How high above its starting point did the protein bar go before it started falling back down? The question of how high the protein bar goes before it starts to come back down is related to a special point on the graph of the function. This point is called the *vertex* of the parabola. What is special about this point?

2. In the first part of the protein bar task you considered three different functions, each one corresponding to a different cliff height. Let's rename the first of these functions as h_1, so that

$$h_1(t) = -16t^2 + 24t + 160.$$

   a. Let $h_2(t)$ denote the height of the protein bar if it is thrown from a cliff that is 56 feet higher. Write the formula for the function h_2.

   b. Let $h_3(t)$ denote the height of the protein bar if it is thrown from a cliff that is 88 feet lower. Write the formula for the function h_3.

   c. **Use technology to graph** all three functions, h_1, h_2, and h_3, on the same axes.

   d. From your graphs, estimate the coordinates of the vertex for each graph.

   e. The t-coordinates of all three vertices are the same. What is the meaning of the t-coordinate in relation to the toss of the protein bar?

   f. The y- coordinate is different for each vertex. Explain the meaning of the y-coordinate for each of the vertices.

3. Consider the formulas for h_1, h_2, and h_3.

   a. How are the formulas different?

   b. Based on your answer to part a, how are the three graphs related? Do you see this relationship in your graphs of the three functions on the same axes? If not, restrict the domain in the viewing window so that you see the part of each graph that corresponds to the same set of t-values.

4. In the introduction above we asked the question: *How high above its starting point did the protein bar go before it started falling back down?*

a. Estimate the answer to the question for the original situation represented by the function h_1.

b. Based on the relationship of the graphs of h_2 and h_3 to h_1, answer the question for the functions h_2 and h_3.

Estimating the vertex from the graphs of our functions allows us to *approximate* how high the protein bar went above its starting point, the question asked in problem 1. However, suppose we need an exact answer to this question. An algebraic method for finding the vertex would give us an exact answer. In the next few problems, we will explore a method for finding the exact coordinates of the vertex of a quadratic function. We will begin with a specific example and then generalize our findings to any quadratic function.

5. Consider the function $f(x) = x^2 - 4x + 9$.

a. Find the *y*- intercept of the graph of this function. Plot this point on graph paper.

b. Find and plot any other points on the graph of *f* that have the same *y*-coordinate as the *y*-intercept.

c. One of the characteristics of a parabola is that the graph has a line of symmetry. Use the points you have plotted so far to find the equation of the line of symmetry for this graph. Draw the line of symmetry as a dotted line on your graph paper.

d. The line of symmetry for a parabola is called the *axis of symmetry*. Explain the relationship between the axis of symmetry and the vertex of a parabola. Then, find the *x*-coordinate of the vertex for the graph of *f*.

e. Find the *y*-coordinate of the vertex for the graph of *f* and state the vertex as a point. Plot this point on your graph paper.

6. Apply steps a – e of problem 6 to find the vertex of each of the functions below.

a. $g(x) = -x^2 - 6x + 7$

b. $f(x) = ax^2 + bx + c$

7. Describe a method for finding the vertex of the graph of any quadratic function given in the form $f(x) = ax^2 + bx + c,\ a \neq 0$.

Mathematics II
The Protein Bar Toss, Part 2
Day 1 Homework

1. Consider the function $f(x) = 4x^2 + 8x - 5$.

   a. Find the y- intercept of the graph of this function. Plot this point on graph paper.

   b. Find and plot any other points on the graph of f that have the same y-coordinate as the y-intercept.

   c. One of the characteristics of a parabola is that the graph has a line of symmetry. Use the points you have plotted so far to find the equation of the line of symmetry for this graph. Draw the line of symmetry as a dotted line on your graph paper.

   d. The line of symmetry for a parabola is called the *axis of symmetry*. Explain the relationship between the axis of symmetry and the vertex of a parabola. Then, find the x-coordinate of the vertex for the graph of f.

   e. Find the y-coordinate of the vertex for the graph of f and state the vertex as a point. Plot this point on your graph paper.

2. Find the vertex of the function $g(x) = 4x^2 – 4x – 15$ using the formula learned in class for the x-coordinate of the vertex of a quadratic function.

3. The height functions h_1, h_2, and h_3, discussed in class, are shown below.

$$h_1(t) = - 16t^2 + 24t + 160$$
$$h_2(t) = - 16t^2 + 24t + 216$$
$$h_3(t) = - 16t^2 + 24t + 72$$

   a. Find the exact coordinates of the vertex of each graph.

   b. For each function, find the exact answer to the question: How high above its starting point did the protein bar go before it started falling back down.

Mathematics II
Task 3: The Protein Bar Toss, Part 2 **Day 2/2**
(GaDOE TE # 9 - 14)

MM2A3. Students will analyze quadratic functions in the forms $f(x) = ax^2 + bx + c$ and $f(x) = a(x - h)^2 + k$.
a. Convert between standard and vertex form.
b. Graph quadratic functions as transformations of the function $f(x) = x^2$.
c. Investigate and explain characteristics of quadratic functions, including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, intervals of increase and decrease, and rates of change.

MM2A4. Students will solve quadratic equations and inequalities in one variable.
c. Solve equations graphically using appropriate technology.
d. Find real and complex solutions of equations by factoring, taking square roots, and applying the quadratic formula.

New vocabulary: vertex form of a quadratic function

Mathematical concepts/skills:
- graphing transformations of $f(x) = x^2$;
- converting between standard and vertex forms of quadratic functions;
- graphing quadratic functions;
- determining the number of x-intercepts of a quadratic function;
- solving quadratic equations by factoring; and
- finding and graphing the x - intercepts of a quadratic function

Prior knowledge:
- multiplying binomials,
- simplifying algebraic expressions, and
- graphing transformations of $f(x) = x^2$, including vertical stretches and shrinks, and vertical shifts

Essential question(s): How can I convert between standard and vertex forms of a quadratic function?

Suggested materials: graphing utility, graph paper

Warm-up: Allow students to compare homework with a partner.

Opening: Discuss the homework, particularly problem 3 (problem 8 in the original GaDOE task). Understanding of this problem will be needed to complete today's lesson.

Worktime: Students should complete problems 8 – 14 of the student task. Problems 12 – 14 may be finished as homework, if time does not allow for these problems to be done in class.

Closing: Problems 8 and 9 provide examples for students to consider in answering problem 10. Problem 10 should be discussed thoroughly during the closing. (See GaDOE teacher notes.)

Homework: Problems 12 – 14 may be finished as homework, if time does not allow for these problems to be done in class.

Differentiated support/enrichment:

Check for Understanding: GaDOE TE *Non-Stop Sports Culminating Task*, Problems 1 and 6.

Resources/materials for Math Support: Students should preview special products studied in Math I, particularly squares of binomials; simplifying algebraic expressions; and writing equations of vertical lines.

Mathematics II
The Protein Bar Toss, Part 2
Day 2 Student Task

8. Consider the function $g(x) = (x - 2)^2 + 5$.

   a. Describe the geometric transformations of the graph of $f(x) = x^2$ that result in the graph of $g(x)$.

   b. Expand the formula of $g(x)$ to the form $g(x) = ax^2 + bx + c$. How do your results compare to the function considered in problem 5?

   c. What did you find to be the vertex of the parabola in problem 5?

9. Consider the function $h(x) = -(x + 3)^2 + 16$.

   a. Describe the geometric transformations of the graph of $f(x) = x^2$ that result in the graph of $h(x)$.

   b. Expand the formula of $h(x)$ to the form $h(x) = ax^2 + bx + c$. How do your results compare to the function in problem 6a?

   c. What did you find to be the vertex of the parabola in problem 6a?

10. For any quadratic function of the form $\boldsymbol{f(x) = ax^2 + bx + c}$:

   a. Explain how you might get a formula for the same function in the form $f(x) = a(x-h)^2 + k$.

   b. What do the h and k in the formula of part a represent relative to the function?

11. Give the **vertex form** of the equations for the functions h_1, h_2, and h_3 and verify algebraically the equivalence with the original formulas for the functions. Remember that you found the vertex for each function in your homework.

12. For the functions given below, put the formula in the vertex form $f(x) = a(x-h)^2 + k$, give the equation of the axis of symmetry, and describe how to transform the graph of $y = x^2$ to create the graph of the given function.

   a. $f(x) = 3x^2 + 12x + 13$

   b. $f(x) = x^2 - 7x + 10$

   c. $f(x) = -2x^2 + 12x - 24$

13. Make a hand-drawn sketch of the graphs of the functions in item 12. Make a dashed line for the axis of symmetry, and plot the vertex, *y*-intercept and the point symmetric with the *y*-intercept.

14. Which of the graphs that you drew in item 14 have *x*-intercepts?

    a. Find the *x*-intercepts that do exist by solving an appropriate equation and then add the corresponding points to your sketch(es) from item 14.

    b. Explain geometrically why some of the graphs have *x*-intercepts and some do not.

    c. Explain how to use the vertex form of a quadratic function to decide whether the graph of the function will or will not have *x*-intercepts. Explain your reasoning.

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Mathematics II: Unit 1

Task 4: Paula's Peaches Revisited

Mathematics II
Task 4: Paula's Peaches Revisited **Day 1/2**
(GaDOE TE # 1 - 3)

MM2A4. Students will solve quadratic equations and inequalities in one variable.
b. Find real and complex solutions of equations by factoring, taking square roots, and applying the quadratic formula.
d. Solve quadratic inequalities both graphically and algebraically, and describe the solutions using linear inequalities.

New vocabulary: quadratic inequality

Mathematical concepts/skills:
- writing quadratic equations to model real situations;
- solving general quadratic equations by factoring; and
- solving quadratic inequalities graphically and algebraically, including plotting points of equality on the number line and testing truth values for the inequality

Prior knowledge:
- writing quadratic equations to model real situations;
- solving simple quadratic equations by factoring; and
- finding points of intersection of functions using a graphing utility

Essential question(s): How can I use quadratic inequalities to model real situations and to answer questions related to those situations?

Suggested materials: graphing calculator, graph paper

Warm-up: Post the following:

Use the information provided to complete as much of the table as possible.

Today we will revisit Paula, the peach grower who wanted to expand her peach orchard last year. In the established part of her orchard, there are 30 trees per acre with an average yield of 600 peaches per tree. Data from the local agricultural experiment station indicated that if Paula chose to plant more than 30 trees per acre in the expanded section of orchard, when the trees reach full production several years from now, the average yield of 600 peaches per tree would decrease by 12 peaches per tree for each tree over 30 per acre.

Let x represents the number of trees Paula might plant per acre in her new section of orchard and Y(x) represents the predicted average yield in peaches per acre. Use the table below to help you write the formula for the function Y.

x (Total number of trees planted per acre)	*Average number of peaches per tree*	*Average yield in peaches per acre*
30	600	18,000
31	588	
32		
33		
34		
x	*(in terms of x)*	$Y(x) =$

Opening: Discuss the scenario above. Good questions to ask might include:

- About how big is an acre? (about the size of a football field)
- Why might the number of peaches per tree decrease if Paula plants more than 30 trees per acre? (To make the point, you might ask students questions like: Would you rather be stuck on an airtight elevator with 5 people or 10 people? Why?)
- What does x represent? (The *total* number of trees planted)

The focus of this lesson is solving quadratic inequalities graphically and algebraically. Students should not spend an inordinate amount of time determining the equation for $Y(x)$. Allow students to share the information they placed in the table and then find the equation for $Y(x)$ as a class.

Worktime: Give students time to complete problem 1. As you monitor their work, encourage students to use what they learned in the previous task about the characteristics of a parabola, including finding the vertex. Even though the domain of the function is limited by the context to the whole numbers between, and including, 30 and 80, the shape is still parabolic and the vertex is included in the limited domain.

Give students time to investigate problem 2. You will probably need to discuss this problem in-depth before allowing students to move on to problem 3. Students should begin by using their calculators to graph the line $y = 18{,}000$ and the function $Y(x)$ on the same axes. Using the *intersect* feature of the calculator, students can see that the graphs intersect at the points (30, 18,000) and (50, 18,000) and that $Y(x)$ is greater than or equal to 18,000 for integer values on the interval $30 \leq x \leq 50$.

The original GaDOE task has been revised in these lesson plans to include the technique of plotting solutions of an equation corresponding to a given inequality on a number line and testing values between these solutions in the inequality for their truth values. Students are then asked to explain this technique in terms of the graph explored in item 2b.

Allow students to complete problem 3 using the techniques discussed in problem 2.

Closing: Allow students to share their work on problem 3. Make sure they discuss the fact that the equation $18000 = 960x - 12x^2$ can be simplified to the equation $x^2 - 80x + 1575 = 0$ as this will facilitate introduction of the next lesson. (See GaDOE teacher notes.)

Homework: The homework below provides more practice in solving quadratic inequalities both graphically and algebraically.

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: Preview should include finding points of intersection of a horizontal line and a quadratic function using a graphing utility; and investigating the relationship between the solution set of a quadratic inequality and the corresponding equation. Students will need extra practice solving quadratic inequalities both graphically and algebraically.

Mathematics II
Paula's Peaches Revisited
Day 1 Student Task

Today we will revisit Paula, the peach grower who wanted to expand her peach orchard last year. In the established part of her orchard, there are 30 trees per acre with an average yield of 600 peaches per tree. Data from the local agricultural experiment station indicated that if Paula chose to plant more than 30 trees per acre in the expanded section of orchard, when the trees reach full production several years from now, the average yield of 600 peaches per tree would decrease by 12 peaches per tree for each tree over 30 per acre.

1. Let x represents the number of trees Paula might plant per acre in her new section of orchard and $Y(x)$ represents the predicted average yield in peaches per acre. Write a formula for the function Y and sketch a graph of the function Y on an appropriate domain.

2. Paula wanted to average at least as many peaches per acre in the new section of orchard as in the established part.

   a. Write an inequality to express the requirement that, for the new section, the average yield of peaches per acre should be *at least* 18,000 peaches, the number in the established section.

   b. Solve the inequality graphically. Your solution should be written as an inequality representing the number of trees planted per acre.

   c. Change your inequality in part 2a to an equation, and solve the equation algebraically. How are the solutions to the equation related to the solution of your inequality?

   d. Plot the solutions of the equation found in part 2c on a number line. Notice that the solutions divide the number line into three subintervals. Choose one value in each subinterval and "test" it in the inequality. Does the value make the inequality true or false? How does this information help you determine a solution to the inequality? How does this technique relate to the graph you drew in part 2b?

3. Suppose that Paula wanted a yield of at least 18900 peaches.

   a. Write an inequality to express the requirement of an average yield of *at least* 18900 peaches per acre.

   b. Solve the inequality graphically.

   c. Solve the inequality algebraically by writing a corresponding equation and using the techniques you learned in problem 2.

   d. Explain how the solution you found in 3c is related to the graph you drew in 3b.

Mathematics II
Paula's Peaches Revisited
Day 1 Homework

Solve each of the following inequalities algebraically. Then illustrate your solution with a graph.

1. $x^2 + x > 12$

2. $-4x^2 + 4x + 3 \geq 0$

3. $2x^2 \leq 5x + 63$

Mathematics I
Task 4: Paula's Peaches Revisited **Day 2/2**
(GaDOE TE # 4 - 7)

MM2A3. Students will analyze quadratic functions in the forms $f(x) = ax^2 + bx + c$ and $f(x) = a(x - h)^2 + k$.
a. Convert between standard and vertex form.
c. Investigate and explain characteristics of quadratic functions, including domain, range, vertex, axis of symmetry, zeros, intercepts, extrema, intervals of increase and decrease, and rates of change.

MM2A4. Students will solve quadratic equations and inequalities in one variable.
b. Find real and complex solutions of equations by factoring, taking square roots, and applying the quadratic formula.
d. Solve quadratic inequalities both graphically and algebraically, and describe the solutions using linear inequalities.

New vocabulary:

Mathematical concepts/skills:
- solving general quadratic equations using the vertex form and extraction of roots;
- investigating quadratic equations with no real roots; and
- solving quadratic inequalities graphically and algebraically

Prior knowledge:
- finding square roots of positive numbers, and
- solving simple quadratic equations by extraction of roots

Essential question(s): How can I use quadratic inequalities to model real situations and to answer questions related to these situations?

Suggested materials: graphing calculator, graph paper

Warm-up: Direct students to compare homework with a partner and be prepared to ask questions about any problems they still do not understand.

Opening: Discuss the homework carefully. Make sure that students understand the relationship between determining solutions of inequalities using test cases and the graphs of the corresponding quadratic functions.

To begin the lesson, remind students of the equation from problem 3: ($x^2 - 80x + 1575 = 0$) discussed in the closing of the previous lesson. Have a whole-class discussion, similar to the one in the beginning of the student task, concerning easier methods of solving quadratic equations.

Worktime: Students should complete problems 4 – 7 of the student task.

Closing: Allow students to share their work. (See GaDOE teacher notes.)

Homework:

Differentiated support/enrichment:

Check for Understanding: GaDOE TE *Non-Stop Sports Culminating Task*, Problem 2.

Resources/materials for Math Support: Preview should include solving simple quadratic equations by extraction of roots.

Mathematics II
Paula's Peaches Revisited
Day 2 Student Task

The equation you solved in part 3c of this task could be written as $x^2 - 80x + 1575 = 0$. Solving by factoring required that you find factors of 1575. With a number as large as 1575, this may have taken you several minutes. Next we explore an alternative method of solving quadratic equations that applies the vertex form of quadratic functions. The advantages of this method are that it can save time over solving equations by factoring when the right factors are hard to find and that it works with equations involving quadratic polynomials that cannot be factored over the integers.

4. Consider the quadratic function $f(x) = x^2 - 80x + 1575$.

   a. What are the x-intercepts of the graph? Explain how you know.

   b. Rewrite the formula for the function so that the x-intercepts are obvious from the formula.

   c. There is a third way to express the formula for the function, the vertex form. Rewrite the formula for the function in vertex form.

   d. Use the vertex form and take a square root to solve for the x-intercepts of the graph. Explain why you should get the same answers as part a.

   e. Explain the relationship between the vertex and the x-intercepts.

5. Consider the quadratic equation $x^2 + 4x - 3 = 0$.

   a. Show that the quadratic polynomial $x^2 + 4x - 3$ cannot be factored over the integers.

   b. Solve the equation by using the vertex form of the related quadratic function and taking a square root.

   c. Approximate the solutions to four decimal places and check them in the original equation.

   d. How are the x-intercepts and axis of symmetry related? Be specific.

6. Suppose that Paula wanted to grow at least 19000 peaches.

   a. Write an inequality for this level of peach production.

   b. Solve the inequality graphically.

c. Solve the corresponding equation algebraically. Approximate any non-integer solutions to four decimal places. Explain how to use the solutions to the equation to solve the inequality.

7. Suppose that Paula wanted to grow at least 20000 peaches.

   a. Write an inequality for this level of peach production.

   b. What happens when you solve the corresponding equation algebraically?

   c. Solve the inequality graphically.

   d. Explain the connection between parts b and c.

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Mathematics II: Unit 1

Task 5: Just the Right Border

Mathematics II
Task 5: Just the Right Border **Day 1/2**
(GaDOE TE # 1 – 3)

MM2A4. Students will solve quadratic equations and inequalities in one variable.
a. Solve equations graphically using appropriate technology.
b. Find real and complex solutions of equations by factoring, taking square roots, and applying the quadratic formula.
c. Analyze the nature of roots using technology and using the discriminant.

New vocabulary: quadratic formula, discriminant

Mathematical concepts/skills:
- writing quadratic equations that model real situations,
- solving general quadratic equations with rational coefficients using the quadratic formula, and
- analyzing the nature of roots of a quadratic equation using the discriminant

Prior knowledge:
- square roots of positive numbers,
- rational versus irrational numbers, and
- simplifying algebraic expressions

Essential question(s): How can I most efficiently solve a quadratic equation?

Suggested materials: graphing calculator, graph paper

Warm-up: Post the following:

Write the given equations in the form $ax^2 + bx + c = 0$. Then identify a, b, and c.

1. $3\left(-2x^2+3\right)=4x$
2. $\frac{x}{3}+\frac{8}{x}=\frac{2}{3}$

Opening: Discuss the warm-up.

Have students read the scenario for problem 1 silently, making notes in the margin of important information. Choose a student to read the scenario aloud and discuss, making sure that all students understand the situation

Worktime: Allow students to work together to complete problems 1a – 1c. Then have a class discussion making sure that all students have a quadratic equation that, when solved, will give the correct value for x.

After discussing problem 1, parts a – c, remind students of the method they used in the previous task for solving quadratic equations that will not factor over the set of integers or, that were difficult to factor. Ask students to help you solve the equation in 1a by writing it in vertex form and taking square roots. Then introduce students to the quadratic formula by having a discussion similar to the one contained in the task following problem 1c. Be sure to define the discriminant carefully.

Allow students to solve the equation from 1a using the quadratic formula and then compare with using the "vertex" method. Which is easier?

Students should complete all parts of problems 2 and 3. Any problems not finished may be finished for homework as long as they are carefully discussed in class before beginning the next lesson.

Closing: Allow students to share their work on problems 2 and 3. (See GaDOE teacher notes.)

Homework:

Differentiated support/enrichment:

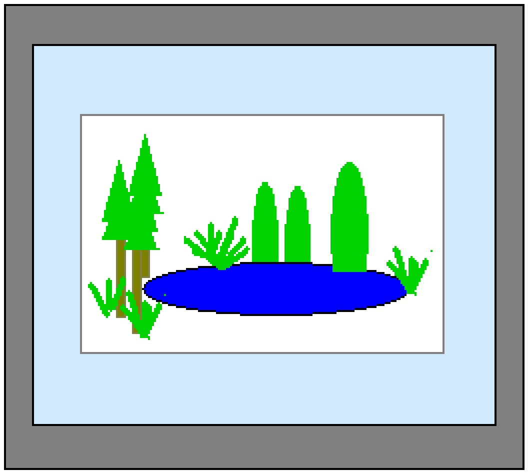
Check for Understanding:

Resources/materials for Math Support: Students should preview writing quadratic equations (similar to those in the warm-up and in the task) in the form $ax^2 + bx + c = 0$ and identifying a, b, and c.

Mathematics II
Just the Right Border
Day 1 Student Task

Just the Right Border Learning Task

1. Hannah is an aspiring artist who enjoys taking nature photographs with her digital camera. Recently she purchased a frame that she thinks would be perfect for framing one of her photographs. The frame is rather large, so the photo needs to be enlarged. Hannah wants to mat the picture. One of her art books suggest that mats should be designed so that the picture takes up 50% of the area inside the frame and the mat covers the other 50%. The dimensions of the area inside of the frame are 20 inches by 32 inches. Hannah wants to enlarge and crop, if necessary, her photograph so that it can be matted with a mat of uniform width and, as recommended, take up 50% of the area inside the mat. See the image at the right.



   a. Let x denote the width of the mat for the picture. Write an equation in x that can be used to find the width of the mat.

   b. Put the equation from part a in the standard form $ax^2 + bx + c = 0$.

   c. Show that the equation from part b cannot be solved by factoring over the integers.

In the previous task, you solved quadratic equations by writing the equation in vertex form and taking square roots. If we apply this method to solve the general equation: $a \bullet x^2 + b \bullet x + c = 0$ for any choice of real number coefficients for *a, b,* and *c* (as long as $a \neq 0$), we find that the solutions have the form $x = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$. This result is called the quadratic formula and is usually stated in summary form as follows.

The Quadratic Formula: If $a \bullet x^2 + b \bullet x + c = 0$ with $a \neq 0$, then

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} .$$

Using the Quadratic Formula is more straightforward and, hence, more efficient than the method of putting the quadratic expression in vertex form and solving by taking square roots.

d. The quadratic formula can be used to solve quadratic equations that cannot be solved by factoring over the set of integers. Using the simplest equivalent equation in standard form, identify *a*, *b,* and *c* from the equation in part *b* and find $b^2 - 4ac$; then substitute these values in the quadratic formula to find the solutions for *x.* Give exact answers for *x* and approximate the solutions to two decimal places.

e. To the nearest tenth of an inch, what should be the width of the mat and the dimensions for the photo enlargement?

2. The quadratic formula can be very useful in solving problems. Thus, it should be practiced enough to develop accuracy in using it and to allow you to commit the formula to memory. Use the quadratic formula to solve each of the following quadratic equations, even if you could solve the equation by other means. Begin by identifying *a*, *b,* and *c* and finding $b^2 - 4ac$; then substitute these values into the formula.

a. $4z^2 + z - 6 = 0$

b. $t^2 + 2t + 8 = 0$

c. $3x^2 + 15x = 12$

d. $25w^2 + 9 = 30w$

e. $7x^2 = 10x$

f. $\frac{t}{2} + \frac{7}{t} = 2$

g. $3\left(2p^2 + 5\right) = 23p$

h. $12z^2 = 90$

i. $\frac{2}{3}q^2 + \frac{1}{4}q = \frac{1}{6}$

3. The expression $b^2 - 4ac$ in the quadratic formula is called the **discriminant** of the quadratic equation in standard form. All of the equations in item 2 had values of a, b, and c that are rational numbers. Answer the following questions for quadratic equations in standard form when ***a*, *b*, and *c* are rational numbers**. Make sure that your answers are consistent with the solutions from item 2.

   a. What is true of the discriminant when there are two real number solutions to a quadratic equation?

   b. What is true of the discriminant when the two real number solutions to a quadratic equation are rational numbers?

   c. What is true of the discriminant when the two real number solutions to a quadratic equation are irrational numbers?

   d. Could a quadratic equation with rational coefficients have one rational solution and one irrational solution? Explain your reasoning.

   e. What is true of the discriminant when there is only one real number solution? What kind of number do you get for the solution?

   f. What is true of the discriminant when there is no real number solution to the equation?

Mathematics II
Task 5: Just the Right Border **Day 2/2**
(GaDOE TE # 4 - 7)

MM2A4. Students will solve quadratic equations and inequalities in one variable.

d. Solve equations graphically using appropriate technology.
e. Find real and complex solutions of equations by factoring, taking square roots, and applying the quadratic formula.
f. Analyze the nature of roots using technology and using the discriminant.

New vocabulary:

Mathematical concepts/skills:

- proof of the quadratic formula;
- using the quadratic formula to solve general quadratic equations with real coefficients, both rational and irrational;
- verifying solutions of quadratic equations graphically; and
- analyzing the nature of roots of a quadratic equation with real coefficients using the discriminant

Prior knowledge:

- square roots of positive numbers,
- simplifying radical expressions,
- rational versus irrational numbers, and
- simplifying algebraic expressions

Essential question(s): How can I most efficiently solve a quadratic equation?

Suggested materials: graphing calculator, graph paper

Warm-up: Direct students to compare homework with a partner.

Opening: Discuss any remaining homework questions.

All students should see a proof of the quadratic formula. Depending on the class, either *show* students a derivation of the formula using item 4 of the student task or allow them to use problem 4 to prove the formula themselves. (See GaDOE teacher notes.) In either case, a whole-class discussion of the proof should occur.

Worktime: Students should complete problems 5 -7 of the student task. It is important that students use a graphing utility to verify their answers for item 5 by finding the x-intercept(s) of the appropriate quadratic function. Not only does this validate the solutions they found but it also reinforces the meaning of the solutions through the use of multiple representations.

Closing: A closing might focus on problem 7, using the problems from item 5 as examples to illustrate student responses to 7a – e. (See GaDOE teacher notes.)

Homework: The homework assigned below consists of problems 8 and 9 of the original DOE task. These problems give students opportunities to apply the concepts they have learned in this task. They should be completed by all students and discussed thoroughly.

Differentiated support/enrichment:

Check for Understanding: GaDOE TE *Non-Stop Sports Culminating Task*, Problems 3 and 8

Resources/materials for Math Support: Students should preview simplifying radical expressions; rational versus irrational numbers; and writing quadratic equations in the form $ax^2 + bx + c = 0$ and identifying a, b and c.

Mathematics II
Just the Right Border
Day 2 Student Task

4. There are many ways to show why the quadratic formula always gives the solution(s) to any quadratic equation with real number coefficients. You can work through one of these by responding to the parts below. Start by assuming that each of *a*, *b*, and *c* is a real number, that $a \neq 0$, and then consider the quadratic equation $ax^2 + bx + c = 0$.

   a. Why do we assume that $a \neq 0$?
   b. Form the corresponding quadratic function, $f(x) = ax^2 + bx + c$, and put the formula for *f*(x) in vertex form, expressing *k* in the vertex form as a single rational expression.

   c. Use the vertex form to solve for *x*-intercepts of the graph and simplify the solution. Hint: Consider two cases, $a > 0$ and $a < 0$, in simplifying $\sqrt{a^2}$.

5. Use the quadratic formula to solve the following equations with real number coefficients. Approximate each real, but irrational, solution correct to hundredths.

   a. $x^2 + \sqrt{5}x + 1 = 0$

   b. $3q^2 - 5q + 2\pi = 0$

   c. $3t^2 + 11 = 2\sqrt{33}t$

   d. $9w^2 = \sqrt{13}w$

6. Verify each answer for item 5 by using a graphing utility to find the *x*-intercept(s) of an appropriate quadratic function.

   a. Put the function for item 5, part c, in vertex form. Use the vertex form to find the *t*- intercept.

   b. Solve the equation from item 5, part d, by factoring.

7. Answer the following questions for quadratic equations in standard from where ***a*, *b*, and *c* are real numbers**.

   a. What is true of the discriminant when there are two real number solutions to a quadratic equation?

   b. Could a quadratic equation with real coefficients have one rational solution and one irrational solution? Explain your reasoning.

   c. What is true of the discriminant when there is only one real number solution?

   d. What is true of the discriminant when there is no real number solution to the equation?

   e. Summarize what you know about the relationship between the discriminant and the solutions of a quadratic of the form $ax^2 + bx + c = 0$ where *a*, *b*, and *c* are real numbers with $a \neq 0$ into a formal statement using biconditionals.

Mathematics II
Just the Right Border
Day 2 Homework

1. A landscape designer included a cloister (a rectangular garden surrounded by a covered walkway on all four sides) in his plans for a new public park. The garden is to be 35 feet by 23 feet and the total area enclosed by the garden and walkway together is 1200 square feet. To the nearest inch, how wide does the walkway need to be?

2. In another area of the park, there will be an open rectangular area of grass surrounded by a flagstone path. If a person cuts across the grass to get from the southeast corner of the area to the northwest corner, the person will walk a distance of 15 yards. If the length of the rectangle is 5 yards more than the width, to the nearest foot, how much distance is saved by cutting across rather than walking along the flagstone path?

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Mathematics II: Unit 1

Task 6: Imagining a New Number

Mathematics II
Task 6: Imagining a New Number **Day 1/2**
(GaDOE TE #3, 4, 6 – 11)

MM2N1. Students will represent and operate with complex numbers.
a. Write square roots of negative numbers in imaginary form.
b. Write complex numbers in the form $a + bi$.
c. Add, subtract, multiply and divide complex numbers.
d. Simplify expressions involving complex numbers.

MM2A4. Students will solve quadratic equations and inequalities in one variable.
b. Find real and complex solutions of equations by factoring, taking square roots, and applying the quadratic formula.
c. Analyze the nature of roots using technology and using the discriminant.

New vocabulary: imaginary number, pure imaginary number, complex number, standard form of a complex number, real part of a complex number, imaginary part of a complex number

Mathematical concepts/skills:
- definition of i and i^2,
- equality of complex numbers,
- computation with complex numbers, and
- determining whether a specific complex number is a solution of a given quadratic equation

Prior knowledge:
- taking square roots of positive numbers;
- operating with radical expressions; and
- adding, subtracting, multiplying, and dividing polynomials

Essential question(s): What are complex numbers and why do I need them?

Suggested materials:

Warm-up: Post the following:

In 1545 the mathematician Gerolamo Cardano posed the problem of dividing ten into two parts whose product is 40. He was using the concept of "divide" in the sense of dividing a line segment of length 10 into two parts of shorter length.

a. Draw a diagram and represent the lengths of the two segments in terms of x.
b. Write an equation in x that expresses the fact that the product of the lengths of the segments is 10.
c. Solve your equation using the quadratic formula. What do you notice?
d. Graph the quadratic function that corresponds to your equation and explain how the graph relates to your results in part c.

Opening: In these plans, the history of square roots of negative numbers has been greatly reduced from the GaDOE task. You might want to share the brief history included here to set the stage for discussing Cardano's problem. Allow students to share their finding for problems $a - d$ of the warm-up before handing out the student task.
(See teacher notes for GaDOE problem #3.)
Allow students time to complete problems 2a and 2b and then discuss as a class.
(See teacher notes for GaDOE problem #4.)

Refer to the task in discussing Euler's use of the notation i for the square root of – 1 and in defining i^2 as -1.

Worktime: Students should begin working the task at problem 3. Monitor student work carefully. Whole class checks for understanding might occur after problems 5 and 7.

Closing: Allow students to present their work. Special attention should be paid to problem 8. (See GaDOE teacher notes, problems 6 – 11.)

Homework: The homework assigned below consists of problems 12 and 13 of the original GaDOE task.

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: Preview should include simplifying and operating with square roots; and adding, subtracting, and multiplying polynomials.

Mathematics II
Imagining a new Number
Day 1 Student Task

In other learning tasks of this unit, you encountered some quadratic equations for which the discriminants were negative numbers. For each of these equations, there is no real number solution because a solution requires that we find the square root of the discriminant and no real number can be the square root of a negative number.

The problem of taking the square root of a negative number was ignored or dismissed as impossible by early mathematicians who encountered it.

According to Eugene W. Hellmich writing in Capsule 76 of *Historical Topics for the Mathematics Classroom, Thirty-first Yearbook of the National Council of Teachers of Mathematics*, 1969:

> The first clear statement of difficulty with the square root of a negative number was given in India by Mahavira (*c*. 850), who wrote: "As in the nature of things, a negative is not a square, it has no square root." Nicolas Chuquet (1484) and Luca Pacioli (1494) in Europe were among those who continued to reject imaginaries.

However, there was a break in the rejection of square roots of negative numbers in 1545 when Gerolamo (or Girolamo) Cardano, known in English as Jerome Cardan, published his important book about algebra, *Ars Magna* (Latin for "The Great Art"). Cardano posed the problem of dividing ten into two parts whose product is 40.

1. Note that, when Cardano stated his problem about dividing ten into two parts, he was using the concept of "divide" in the sense of dividing a line segment of length 10 into two parts of shorter length.

    a. Show that Cardano's problem leads to the quadratic equation $x^2 - 10x + 40 = 0$.

    b. Find the solutions to this equation given by the quadratic formula even though they are not real numbers.

2. Rather than reject the solutions to $x^2 - 10x + 40 = 0$ as impossible, Cardano simplified them to obtain $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$, stated that these solutions were "manifestly impossible", but plunged ahead by saying "nevertheless, we will operate." He "operated" by treating these expressions as numbers that follow standard rules of algebra and checked that they satisfied his original problem.

    a. Assuming that these numbers follow the usual rules of algebra, verify that their sum is 10.
    b. Also assuming that $\left(\sqrt{-15}\right)\left(\sqrt{-15}\right) = -15$, verify that the product of the numbers is 40.

So, Cardano was the first to imagine that there might be some numbers in addition to the real numbers that we represent as directed lengths. However, Cardano did not pursue this idea. According to Hellmich in his mathematics history capsule, "Cardano concludes by saying that these quantities are 'truly sophisticated' and that to continue working with them would be 'as subtle as it would be useless.'" Cardano did not see any reason to continue working with the numbers because he was unable to see any physical interpretation for numbers. However, other mathematicians saw that they gave useful algebraic results and continued the development of what today we call ***complex numbers***.

Rene Descartes, the French mathematician who gave us the Cartesian coordinate system for plotting points, did not see a geometric interpretation for the square root of a negative number so in his book *La Geometrie* (1637) he called such a number "imaginary." This term stuck so that we still refer to the square root of a negative number as "imaginary." By the way, Descartes is also the one who coined the term "real" for the real numbers.

In 1748, Leonard Euler, one of the greatest mathematicians of all times, started the use of the notation "*i*" to represent the square root of – 1, that is, $i = \sqrt{-1}$. Thus,

$$\begin{aligned} i^2 &= \left(\sqrt{-1}\right)\left(\sqrt{-1}\right) \\ &= \sqrt{-1}\sqrt{-1} = \frac{n\left(n-1\right)}{2} \\ &= -1, \end{aligned}$$

since *i* represents the number whose square is – 1.

This definition preserves the idea that the square of a square root returns us to the original number, but also shows that one of the basic rules for working with square roots of real numbers,

$$\textit{for any real numbers a and b,} \left(\sqrt{a}\right)\left(\sqrt{b}\right) = \sqrt{a}\sqrt{b} = \sqrt{ab}$$

does **<u>not</u>** hold for square roots of negative numbers because, if that rule were applied we would not get -1 for i^2.

3. As we have seen, the number *i* is not a real number; it is a new number. We want to use it to expand from the real numbers to a larger system of numbers.

   a. What meaning could we give to $2i, 3i, 4i, 5i, \ldots$?

   b. Find the square of each of the following: $2i,\ 7i,\ 10i,\ 25i$.

c. How could we use i to write an expression for each of the following: $\sqrt{-4}, \sqrt{-25}, \sqrt{-49}$?

d. What meaning could we give $-i, -2i, -3i, -4i, -5i, \ldots$?

e. Write an expression involving i for each of the following: $-\sqrt{-9}, -\sqrt{-16}, -\sqrt{-81}$.

An ***imaginary number*** is any number that can be written in the form $b\,i$, where b is an real number and $i = \sqrt{-1}$. Imaginary numbers are also sometimes called ***pure*** imaginary numbers.

4. Write each of the following imaginary numbers in the standard form *bi.*

$$\sqrt{-\frac{1}{36}},\ \sqrt{-11},\ \sqrt{\frac{-5}{64}},\ -\sqrt{-7},\ -\sqrt{-18}.$$

Any number that can be written in the form $a + bi$, where a and b are real numbers, is called a ***complex number***. We refer to the form $a + bi$ as the ***standard form*** of a complex number and call a the ***real part*** and b the ***imaginary part***.

5. Write each of the following as a complex number in standard form and state its real part and its imaginary part.

$$6 - \sqrt{-1},\ -12 + \sqrt{-100},\ 31 - \sqrt{-20}$$

The set of complex numbers includes all of the real numbers (when the imaginary part is 0), all of the imaginary numbers (when the real part is 0), and lots of other numbers that have nonzero real and imaginary parts.

We say that two complex numbers are ***equal*** if their real parts are equal and their imaginary parts are equal, that is,

if $a + bi$ and $c + di$ are complex numbers,
then $a + bi = c + d\,i$ if and only if $a = c$ and $b = d$.

In order to define operations on the set of complex numbers in a way that is consistent with the established operations for real numbers, we define addition and subtraction by combining the real and imaginary parts separately:

if $a + bi$ and $c + di$ are complex numbers,
then $\boldsymbol{(a + bi) + (c + d\,i) = (a + c) + (b + d)i}$.

$\boldsymbol{(a + bi) - (c + d\,i) = (a - c) + (b - d)i}$

6. Apply the above definitions to perform the indicated operations and write the answers in standard form.

   a. $(3+5i)+(2-6i)$

   b. $(5-4i)-(-3+5i)$

   c. $13i-(3-i)$

   d. $(-7-9i)-(-2)$

   e. $(5.4+8.3i)+(-3.7+4.6i)$

   f. $\left(\frac{3}{7}-\frac{5}{7}i\right)-\left(\frac{4}{7}+\frac{4}{7}i\right)$

To multiply complex numbers, we use the standard form of complex numbers, multiply the expressions as if the symbol i were an unknown constant, and then use the fact that $i^2=-1$ to continue simplifying and write the answer as a complex number in standard form.

7. Perform each of the following indicated multiplications, and write your answer as a complex number in standard form.

   a. $6i(2-i)$

   b. $(2-i)(3+4i)$

   c. $\frac{2}{7}(9-6i)$

   d. $\left(4-i\sqrt{13}\right)\left(2+i\sqrt{13}\right)$ -- Note writing i in front of an imaginary part expressed as a root is standard practice to make the expression easier to read.

   e. $\left(i\sqrt{5}\right)\left(i\sqrt{5}\right)$

   f. $\left(\sqrt{-6}\right)\left(-\sqrt{-12}\right)$

8. For each of the following, use substitution to determine whether the complex number is a solution to the given quadratic equation.

   a. Is $-2+3i$ a solution of $x^2+4x+13=0$? Is $-2-3i$ a solution of this same equation?

   b. Is $2+i$ a solution of $x^2-3x+3=0$? Is $2-i$ a solution of this same equation?

Mathematics II
Imagining a New Number
Day 1 Homework

1. Find each of the following products.

   a. $(1+i)(1-i)$

   b. $(5-2i)(5+2i)$

   c. $\left(\sqrt{7}+8i\right)\left(\sqrt{7}-8i\right)$

   d. $\left(-\frac{1}{2}-\frac{2}{3}i\right)\left(-\frac{1}{2}+\frac{2}{3}i\right)$

   e. $\left(4+i\sqrt{6}\right)\left(4-i\sqrt{6}\right)$

2. The complex numbers $a+bi$ and $a-bi$ are called ***complex conjugates***. Each number is considered to be the complex conjugate of the other.

   a. Review the results of your multiplications in item 1, and make a general statement that applies to the product of any complex conjugates.

   b. Make a specific statement about the solutions to a quadratic equation in standard form when the discriminant is a negative number.

Mathematics II
Task 6: Imagining a New Number **Day 2/2**
(GaDOE TE #12 - 20)

MM2N1. Students will represent and operate with complex numbers.
a. Write square roots of negative numbers in imaginary form.
b. Write complex numbers in the form $a + bi$.
c. Add, subtract, multiply and divide complex numbers.
d. Simplify expressions involving complex numbers.

MM2A4. Students will solve quadratic equations and inequalities in one variable.
b. Find real and complex solutions of equations by factoring, taking square roots, and applying the quadratic formula.
c. Analyze the nature of roots using technology and using the discriminant.

New vocabulary: complex conjugates, complex number plane, absolute value of a complex number

Mathematical concepts/skills:
- multiplication of complex conjugates,
- division of complex numbers,
- graphing in the complex plane,
- finding absolute values of complex numbers,
- simplifying powers of i,
- solving general quadratic equations with complex solutions, and
- analyzing the nature of roots using the discriminant

Prior knowledge:
- adding, subtracting, multiplying, and dividing polynomials; and
- graphing in the Cartesian coordinate plane

Essential question(s): What are complex numbers and why do I need them?

Suggested materials: graphing utility, graph paper

Warm-up: Direct students to compare homework with a partner and be prepared to ask questions related to the problems they still do not understand.

Opening: Discuss any questions related to homework. Problem 10 b should be discussed thoroughly.

Worktime: After students have had time to complete problems 9 and 10 of the task, you may want to have a whole-class check for understanding of computing with complex numbers before moving on to graphing in the complex plane.

This is a good time to have a discussion of the complex number system. (See page 109 of the GaDOE teacher notes.) You may begin this discussion by asking students to draw a flowchart or Venn diagram representing the hierarchal relationships between the types of numbers they have studied so far. Sets of numbers you may list for them to include in the diagram are *real, rational, irrational, integers, whole numbers, counting numbers, pure imaginary numbers*, and *complex numbers* in the form $a + bi$, where neither a nor b are 0.

Closing: Allow students to present their work on selected problems from items 9 - 15. (See GaDOE teacher notes, problems 14 - 20.)

Homework: The problems listed as homework for this section give students opportunities to find complex solutions of quadratic equations. They should be completed and discussed. Students may need additional practice in finding complex solutions.

Differentiated support/enrichment:

Check for Understanding:

- GaDOE TE *Non-Stop Sports Culminating Task*, Problem 5.

- The problem below requires students to summarize the information they have studied related to solutions of a quadratic equation given the nature of the discriminant.

Write a complete statement to describe the solutions of a quadratic equation in the form $ax^2 + bx + c = 0$, *where* $a \neq 0$, *when the discriminant* $b^2 - 4ac$ *has the following values:*

- $b^2 - 4ac > 0$
- $b^2 - 4ac = 0$
- $b^2 - 4ac < 0$

Resources/materials for Math Support: Preview should include operating with polynomials.

Mathematics II
Imagining a new Number
Day 2 Student Task

So far in our work with operations on complex numbers, we have discussed addition, subtraction, and multiplication and have seen that, when we start with two complex numbers in standard form and apply one of these operations, the result is a complex number that can be written in standard.

9. Now we come to division of complex numbers. Consider the following quotient: $\frac{-2+3i}{3-4i}$.

   This expression is not written as a complex number in standard form; in fact, it is not even clear that it can be written in standard form. The concept of complex conjugates is the key to carrying out the computation to obtain a complex number in standard form.

   a. Find the complex conjugate of $3-4i$, which is the denominator of $\frac{-2+3i}{3-4i}$.

   b. Multiply the numerator and denominator of $\frac{-2+3i}{3-4i}$ by the complex conjugate from part a.

   c. Explain why $\frac{-2+3i}{3-4i} = \frac{-2+3i}{3-4i} \cdot \frac{3+4i}{3+4i}$ and then use your answers from part b to obtain $-\frac{18}{25}+\frac{1}{25}i$ as the complex number in standard form equal to the original quotient.

   d. For real numbers, if we multiply the quotient by the divisor we obtain the dividend. Does this relationship hold for the calculation above?

10. Use the same steps as in item 9 to simplify each of the following quotients and give the answer as a complex number in standard form.

    a. $(5-6i)\div(2+i)=\frac{5-6i}{2+i}$

    b. $\frac{3\sqrt{2}}{7-i\sqrt{2}}$

    c. $\frac{14+2i}{2i}$

    d. $\frac{2+\sqrt{-8}}{-5+\sqrt{-18}}$

In our discussion of the early history of the development of complex numbers, we noted that Cardano did not continue work on complex numbers because he could not envision any geometric interpretation for them. It was over two hundred years until, in 1797, the Norwegian surveyor Caspar Wessel presented his ideas for a very simple geometric model of the complex numbers. For the remainder of the task you will investigate the modern geometric representation of complex numbers, based on Wessel's work and that of Wallis, Argand, Gauss, and other mathematicians, some of whom developed the same interpretation as Wessel independently.

In representing the complex numbers geometrically, we begin with a number line to represent the pure imaginary numbers and then place this number line perpendicular to a number line for the real numbers. The number $0i = 0$ is both imaginary and real, so it should be on both number lines. Therefore, the two number lines are drawn perpendicular and intersecting at 0, just as we do in our standard coordinate system. Geometrically, we represent the complex number $a + bi$ by the point (a, b) in the coordinate system. When we use this representation, we refer to a complex number as a point in the ***complex number plane.*** Note that this is a very different interpretation for points in a plane that the one we use for graphing functions whose domain and range are subsets of the real numbers as we did in considering cubic functions in item 5.

11. Use a complex number plane to graph and label each of the following complex numbers: $2+3i$, $-8-6i$, $-5i$, -2, $7-3i$, $6i$, $-3+4i$, 3.5

12. Geometrically, what is the meaning of the absolute value of a real number? We extend this idea and define the ***absolute value of the complex number*** $\boldsymbol{a+bi}$ to be the distance in the complex number plane from the number to zero. We use the same absolute value symbol as we did with real numbers so that $|a+bi|$ represents the absolute value of the complex number $a+bi$.

    a. Find the absolute value of each of the complex numbers plotted in item 11.

    b. Verify your calculations from part a geometrically.

    c. Write a formula in terms of a and b for calculating $|a+bi|$.

    d. What is the relationship between the absolute value of a complex number and the absolute value of its conjugate? Explain.

    e. What is the relationship between the absolute value of a complex number and the product of that number with its conjugate? Explain.

13. The relationship you described in part e of item 12 is a specific case of a more general relationship involving absolute value and products of complex numbers. Find each of the following products of complex numbers from item 11 and compare the absolute value of each product to the absolute values of the factors in the product. (Note that you found the length of each of the factors below in item 12, part a. Finally, make and prove a conjecture about the absolute value of the product of two complex numbers.

   a. $(2+3i)(7-3i)$

   b. $(-3+4i)(5i)$

   c. $(-2)(-8-6i)$

   d. $(6i)(3.5)$

14. By definition, $i^1 = i$ and $i^2 = -1$.

   a. Find $i^3,\ i^4,\ i^5,\ \ldots,\ i^{12}$. What pattern do you observe?

   b. How do the results of your calculations of powers of i relate to item 13?

   c. Devise a way to find any positive integer power of i, and use it to find the following powers of i: i^{26}, i^{55}, i^{136}, i^{373}.

15. We conclude this task with an exploration of the geometry of multiplying a complex number by i.

   a. Graph $i,\ i^2,\ i^3,\ i^4$ in the complex number plane. How does the point move each time it is multiplied by i?

   b. Let $z = 5+12i$. What is iz?

   c. Plot each of the complex numbers in part b and draw the line segment connecting each point to the origin.

   d. Explain why multiplying by i does not change the absolute value of a complex number.

   e. What geometric effect does multiplying by i seem to have on a complex number? Verify your answer for the number z from part b.

Mathematics II
Imagining a new Number
Day 2 Homework

Solve each of the following quadratic equations using the quadratic formula.

1. $x^2 - 2x + 5 = 0$
2. $x^2 + 7 = 0$
3. $2x^2 = 6x - 7$

Mathematics II
Task 7: Geometric Connections **Day 1/2**
(GaDOE TE #1 - 4)

MM2A3. Students will analyze quadratic functions in the forms $f(x) = ax^2 + bx + c$ and $f(x) = a(x - h)^2 + k$.
d. Explore arithmetic sequences and various ways of computing their sums.
e. Explore sequences of partial sums of arithmetic series as examples of quadratic functions.

MM2A4. Students will solve quadratic equations and inequalities in one variable.
b. Find real and complex solutions of equations by factoring, taking square roots, and applying the quadratic formula.

New vocabulary: vertex edge graphs

Mathematical concepts/skills:
- constructing vertex-edge graphs;
- writing recursive definitions for sequences;
- viewing the n^{th} term of a sequence, whose second differences are constant, as a partial sum of an arithmetic series;
- writing closed form formulas the n^{th} term of a sequence of partial sums as a quadratic expression; determining the number of pairs of objects that can be formed from n objects (number of combinations of n objects taken two at a time); and
- solving real problems using quadratic equations resulting from formulas for sequences of partial sums

Prior knowledge:
- subscript notation for terms in a sequence,
- recursive and closed definitions for arithmetic sequences, and
- writing sequences as functions with whole number domains

Essential question(s): How many different pairs of objects can be form from n objects?

Suggested materials: calculators, grid paper, square tiles

Warm-up: Post the following:

Write a recursive definition for the following sequence:

1, 5, 11, 19, 29, …

Represent the following sequence using function notation:

1, 6, 11, 16, …

Opening: Discuss the warm-up. The first item reviews notation and recursive definitions for sequences. Both concepts were studied in grade 8. The second problem reviews the concept of writing a closed form definition of an arithmetic series, studied in grade 8, and the idea of writing sequences as functions with whole number domains, studied in Math I.

A recursive definition of the first sequence can be written as $t_n = t_{(n-1)} + 2n,\ t_1 = 1$.
The second sequence can be represented as $f(n) = 5n - 4$, where n is a positive integer.
Post the three scenarios given at the beginning of the task and choose students to read the situations aloud. Ask if they think the situations have anything in common. If so, what? If not, why not?

Worktime: Hand out the task. Monitor student work carefully. Ask guiding questions to redirect students who are "off track". Problem 2f is essential in understanding the purpose of the entire task. You may want to stop after students have had time to complete problem 2 and have a whole-class discussion of the work to this point. Student responses to 2f should include all of the following information. (See GaDOE teacher notes.)

Since $p_1 = 0$, we apply the recursive definition repeatedly to obtain that

$$\begin{aligned} p_2 &= p_1 + 1 = 0 + 1 = 1, \\ p_3 &= p_2 + 2 = 1 + 2, \\ p_4 &= p_3 + 3 = 1 + 2 + 3, \\ &\vdots \end{aligned}$$

Continuing this pattern, we see that $p_n = 1 + 2 + \cdots + (n-1)$.

Another whole-class discussion, after problem 3, may be useful in making sure that all students have developed a closed form formula for p_n (item 3c) before moving to applications of this formula in problem 4.

Closing: Allow students to present their work. Problem 5 gives students an opportunity to compare the formula they found for p_n in item 3c to a formula with which they should already be familiar. Students studied combinations in Math I and should remember that the number of combinations of n objects taken r at the time is calculated as $C_r^n = \dfrac{n!}{r!(n-r)!}$. Therefore,

$$\begin{aligned} C_2^n &= \frac{n!}{2!(n-2)!} \\ &= \frac{n(n-1)(n-2)!}{2(n-2)!} \\ &= \frac{n(n-1)}{2}. \end{aligned}$$

Homework: Problem 4 may be assigned as homework as long as students have derived the formula for p_n and have an understanding of how it can be applied to the situations described.

Differentiated support/enrichment:

Check for Understanding:

Resources/materials for Math Support: Students should preview subscript notation for terms of a sequence, writing recursive definitions for a variety of sequences, finding closed form formulas for arithmetic sequences, and writing sequences as functions with whole number domains.

Mathematics II
Geometric Connections
Day 1 Student Task

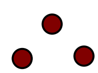
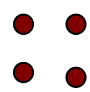
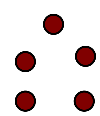
Scenario 1: A group of college freshmen attend a freshmen orientation session. Each is given a numbered "Hello" nametag. Students are told to shake hands with every other student there, and they do. How many handshakes are exchanged?

Scenario 2: An airline has several hub cities and flies daily non-stop flights between each pair of these cities. How many different non-stop routes are there?

Scenario 3: A research lab has several computers that share processing of important data. To insure against interruptions of communication, each computer is connected directly to each of the other computers. How many computer connections are there?

The questions asked at the end of the scenarios are really three specific versions of the same purely mathematical question: Given a set of objects, how many different pairs of objects can we form? One of the easiest ways to approach this problem involves thinking geometrically. The objects, whether they are college freshmen, cities, computers, etc., can be represented as points, and each pairing can be represented by drawing a line segment between the points. Such a representation is called a ***vertex-edge graph***. You will explore vertex-edge graphs further in Mathematics III, but for now our focus is counting pairs of objects.

1. Use a vertex-edge graph and an actual count of all possible edges between pairs of points to fill in the table below. When there are more than two vertices, it helps to arrange the points as if they are vertices of regular polygons as shown in the table.

Number of points/vertices	1	2	3	4	5
Example diagram of vertices					
Number of line segments/edges					

2. Our goal is to find a formula that gives the number of pairs of objects as a function of n, the number of objects to be paired. Before we try to find the general formula, let's find the answers for a few more specific values.

   a. Draw a set of 6 points and all possible edges among 5 of the points. How many edges do you have so far? How many additional edges do you draw to complete the diagram to include all possible edges between two points? What is the total number of edges?

   b. Draw a set of 7 points and all possible edges among 6 of the points. How many edges do you have so far? How many additional edges do you draw to complete the diagram to include all possible edges between two points? What is the total number of edges?

c. We are trying to find a formula for the number of pairs of objects as a function of the number of objects. What is the domain of this function? Why does this domain allow us to think of the function as a sequence?

d. Denote the number of pairs of n objects by p_n, so that the sequence is $p_1, p_2, p_3, \ldots$. Remember that a recursive sequence is one that is defined by giving the value of at least one beginning term and then giving a recursive relation that states how to calculate the value of later terms based on the value(s) of one or more earlier terms. Generalize the line of reasoning from parts a and b to write a recursive definition of the sequence $p_1, p_2, p_3, \ldots$. Which beginning term(s) do you need to specify explicitly? What is the recursive relation to use for later terms?

e. Use your recursive definition to complete the table below and verify that it agrees with the previously found values.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
p_n															

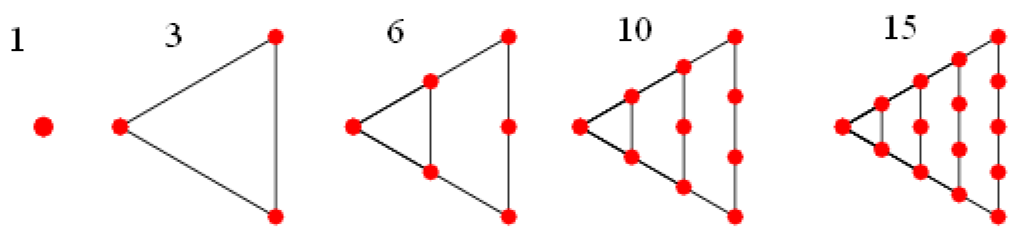
f. Explain how the recursive definition of $p_1, p_2, p_3, \ldots$ also leads to expressing p_n as a sum of integers by completing the pattern started below.

3. The numbers in the sequence $p_2, p_3, p_4 \ldots$ are known as the *triangular numbers*, so called because that number of dots can be arranged in a triangular pattern as shown below. (Image from Weisstein, Eric W. "Triangular Number." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/TriangularNumber.html

a. The standard notation for the triangular numbers is $T_1, T_2, T_3, \ldots$. Through exploration of patterns and/or geometric representations of these numbers, find a closed form formula for the n^{th} triangular number, T_n.

b. Explain the relationship between the sequences $p_1, p_2, p_3, \ldots$ and $T_1, T_2, T_3, \ldots$.

c. Use the relationship explained in part b to write a closed form formula for p_n



4. Now we return to the scenarios.

   a. In Scenario 1, if there are 40 students at the orientation session, how many handshakes are exchanged?

   b. In Scenario 2, if there are 190 airline routes between pairs of cities, how many cities are there in the group of hub cities?

   c. In Scenario 3, if there are 45 computer connections, how many computers does the research lab use?

5. The formula you found in item 3, part c, and then applied in answering the questions of item 4, is well-known as the formula for counting ***combinations of n objects taken two at a time.*** Use what you learned about combinations in Math I to justify that the two formulas are the same.

Mathematics II
Task 7: Geometric Connections **Day 2/2**
(GaDOE TE #9 - 11)

MM2A3. Students will analyze quadratic functions in the forms $f(x) = ax^2 + bx + c$ and $f(x) = a(x - h)^2 + k$.
d. Explore arithmetic sequences and various ways of computing their sums.
e. Explore sequences of partial sums of arithmetic series as examples of quadratic functions.

New vocabulary: arithmetic series

Mathematical concepts/skills:

- finding closed form formulas for an arithmetic series s_n, and
- exploring various ways of computing sums of arithmetic series

Prior knowledge:

- subscript notation for terms in a sequence,
- recursive and closed definitions for arithmetic sequences, and
- sequences as functions with whole number domains

Essential question(s): How can I find the sum of an arithmetic series?

Suggested materials: calculators

Warm-up: If any part of problem 4 of the previous lesson was assigned as homework, allow students to compare their solutions.

Post the following:

1. *Represent the n^{th} triangular number as a sum of terms of an arithmetic sequence.*
2. *State the closed form formula for the n^{th} triangular number.*
3. *Use your formula to find T_{20}.*

Opening: Discuss any remaining concerns related to problem 4 of the previous lesson.

The warm-up is designed to review the work done in the previous lesson. Answers are as follows:

1. $T_n = 1 + 2 + 3 + 4 + \ldots + n$

2. $T_n = \dfrac{n(n+1)}{2}$

3. $T_{20} = 210$

Worktime: Students should complete problems 6 – 8 of the task. Allow students plenty of time, in problem 8, to develop their own methods for finding the sum of an arithmetic series.

Closing: Allow students to share the various methods they developed for finding the sum of a series. Compare various students methods to each other and to the formula derived in problem 7. (See GaDOE teacher notes, problems 9 -11.)

Homework: The homework assigned below is problem 12 of the original GaDOE task. Students should have an opportunity to complete this problem, discuss their ideas, and check their work for accuracy.

Differentiated support/enrichment:

Check for Understanding:

- GaDOE TE *Non-Stop Sports Culminating Task*, Problems 4 and 7.

- The problem below is problem 13 of the original GaDOE task. It provides an opportunity for students to apply sums of arithmetic series to a real world situation and should be completed by all students.

In 2001, the owner of two successful restaurants in south metropolitan Atlanta developed a long term expansion plan that led to the opening of five more restaurants in 2002. In 2003, eight new restaurants opened; in 2004, eleven new restaurants opened. Assume that the owner continued with this plan of increasing the number of new restaurants by three each year.

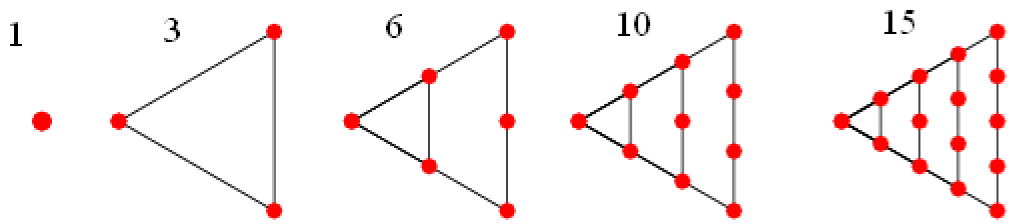
a. How many restaurants did the owner have in operation by the end of 2007?

b. If the owner continues with this plan, in what year will the 500th *restaurant open?*

Resources/materials for Math Support: Students should preview subscript notation for terms of a sequence, writing recursive definitions for a variety of sequences, finding closed form formulas for arithmetic sequences, and writing sequences as functions with whole number domains.

Mathematics II
Geometric Connections
Day 2 Student Task

In item 3, we found that summing the terms of the sequence 1, 2, 3, … gives the *triangular numbers*, so called because that number of dots can be arranged in a triangular pattern as shown below. (Image from Weisstein, Eric W. "Triangular Number." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/TriangularNumber.html)



The sequence 1, 2, 3, … starts with 1 and adds 1 each time. Next we consider what happens if we start with 1 and add 2 each time.

6. The sequence that starts with 1 and adds 2 each time is very familiar sequence.

   a. List the first five terms of the sequence and give its common mathematical name.

   b. Why is the sequence an arithmetic sequence?

   c. Complete the table below for summing terms of this sequence.

Number of terms to be summed	Indicated sum of terms	Value of the sum
1	1	1
2	1 + 3	
3		
4		
5		
6		
n		

   d. Give a geometric interpretation for the sums in the table.

   e. Fill in the last row of the table above when the number of terms to be summed is n, including a conjecture about the value for the last column. Use ". . ." in the indicated sum but include an expression, in terms of n, for the last term to be summed.

Our next goal is justification of the conjecture you just made in item 6, part e. However, rather than work on this specific problem, it is just as easy to consider summing any arithmetic sequence. The sum of the terms of a sequence is called a series so, as we proceed, we'll be exploring closed form formulas for ***arithmetic series.***

7. Let a_1 and d be real numbers and let $a_1, a_2, a_3, \ldots$ be the arithmetic sequence with first term a_1 and common difference d. We'll use some standard notation for distinguishing between a sequence and the corresponding series which sums the terms of the sequence.

   a. Fill in the table below with expressions in terms of a_1 and d.

Term of the sequence	Expression using a_1 and d
a_1	
a_2	
a_3	
a_4	
a_5	
a_6	

   b. Let $s_1, s_2, s_3, \ldots$ be the sequence defined by the following pattern:

$$s_1 = a_1$$
$$s_2 = a_1 + a_2$$
$$s_3 = a_1 + a_2 + a_3$$
$$s_4 = a_1 + a_2 + a_3 + a_4$$
$$\vdots$$
$$s_n = a_1 + a_2 + \cdots + a_n$$

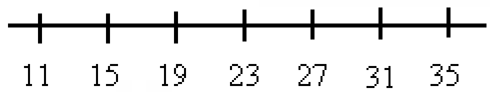
   Conjecture and prove a closed form formula for the arithmetic series s_n.

   c. Use the sum of an arithmetic series formula to verify the conjecture for the sum of the arithmetic series in item 6.

8. Next we'll explore a different approach to a closed form formula for arithmetic series, one that is quite useful when we are examining the type of series that occurs in a problem like the following. (This item is adapted from the "Common Differences" Sample Secondary Task related to K-12 Mathematics Benchmarks of The American Diploma Project, copyright 2007, Charles A. Dana Center at the University of Texas at Austin.)

Sam plays in the marching band and is participating in the fund raiser to finance the spring trip. He's taking orders for boxes of grapefruit to be delivered from a Florida grove just in time for the Thanksgiving and Christmas holiday season. With the help of his parents and grandparents, he is trying to win the prize for getting the most orders. The number of orders he got for each day of the last week are, respectively, 11, 15, 19, 23, 27, 31, 35. What is the total number of orders?

a. The total number of orders is the arithmetic series $11+15+19+23+27+31+35$. The numbers in the corresponding arithmetic sequence are plotted on a number line below.



Find a shortcut for finding the sum of these seven terms. State your shortcut as a mathematical conjecture, and give a justification that your conjecture works.

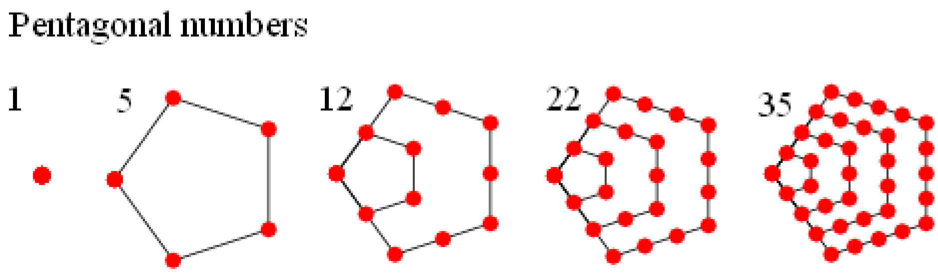
b. Test your conjecture from part a on an arithmetic series with exactly 5 terms. Does it work for a series with exactly 9 terms? Does your conjecture apply to a series with exactly 6 terms?

c. Modify your conjecture, if necessary, so that it describes a general method for finding any finite sum of consecutive terms from an arithmetic sequence. Give a justification that your method will always work.

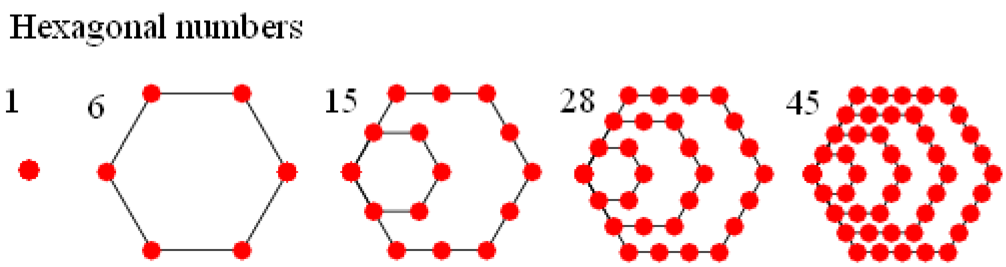
d. How does your method compare to the formula found in problem 7?

Mathematics II
Geometric Connections
Day 2 Homework

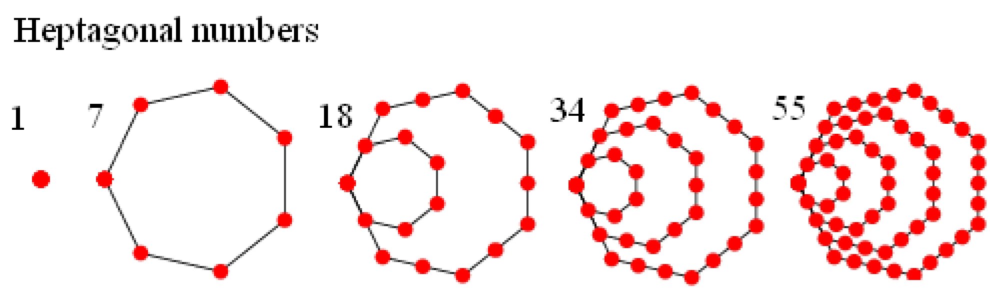
In addition to the well known triangular and square numbers, there are ***polygonal numbers*** corresponding to each type of polygon. The names relate to the figures that can be made from that number of dots. For example, the first five pentagonal, hexagonal, and heptagonal numbers are shown below.



(Image from Weisstein, Eric W. "Pentagonal Number." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/PentagonalNumber.html)



(Image from Weisstein, Eric W. "Hexagonal Number." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/HexagonalNumber.html)



(Image from Weisstein, Eric W. "Heptagonal Number." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/HeptagonalNumber.html)

13. As shown above, the number 1 is considered to be a polygonal number for each size polygon. The actual numbers in the sequence of each type of polygonal numbers are sums of an arithmetic series; for example, the n^{th} triangular number is the sum of the arithmetic series $1+2+\cdots+n$. Use your knowledge of arithmetic sequences and series and of polygonal numbers complete the table below.

Polygonal number name	First five terms of the sequence of such numbers	Arithmetic sequence whose terms sum to form the numbers	Values a_1, d	Formula for n^{th} polygonal number
triangular				
square				
pentagonal				
hexagonal				
heptagonal				
octagonal				