ATLANTA PUBLIC SCHOOLS
Mathematics & Science Initiative
Making A Difference

Atlanta Public Schools

Teacher's Curriculum Supplement

Common Core Georgia Performance Standards Mathematics III

Unit 3: Exponential and Logarithmic Functions

GE Foundation

This document has been made possible by funding from the GE Foundation Developing Futures grant, in partnership with Atlanta Public Schools. It is derived from the Georgia Department of Education Math III Framework and includes contributions from Georgia teachers. It is intended to serve as a companion to the GA DOE Math III Framework Teacher Edition.

Preface

We are pleased to provide this supplement to the Georgia Department of Education's Mathematics III Framework. It has been written in the hope that it will assist teachers in the planning and delivery of the new curriculum, particularly in these first years of implementation. This document should be used with the following considerations.

- The importance of working the tasks used in these lessons cannot be overstated. In planning for the teaching of the Georgia Performance Standards in Mathematics teachers should work the tasks, read the teacher notes provided in the Georgia Department of Education's Mathematics III Framework Teacher Edition found at georgiastandards.org in *Learning Village*, and *then* examine the lessons provided here.

- This guide provides day-by-day lesson plans. While a detailed scope and sequence and established lessons may help in the implementation of a new and more rigorous curriculum, it is hoped that teachers will assess their students informally on an on-going basis and use the results of these assessments to determine (or modify) what happens in the classroom from one day to the next. Planning based on student need is much more effective than following a pre-determined timeline.

- It is important to remember that the Georgia Performance Standards provide a balance of concepts, skills, and problem solving. Although this document is primarily based on the tasks of the Framework, we have attempted to help teachers achieve this all important balance by embedding necessary skills in the lessons and including skills in specific or suggested homework assignments. The teachers and writers who developed these lessons, however, are not in your classrooms. It is incumbent upon the classroom teacher to assess the skill level of students on every topic addressed in the standards and provide the opportunities needed to master those skills.

- In most of the lesson templates, the sections labeled *Differentiated support/enrichment* have been left blank. This is a result of several factors, the most significant of which was time. It is hoped that as teachers use these lessons, they will contribute their own ideas, not only in the areas of differentiation and enrichment, but in other areas as well. Materials and resources abound that can be used to contribute to the teaching of the standards.

On the topic of differentiation, it is critical to reiterate that many of the strategies used in a standards-based mathematics classroom promote differentiation. These strategies include

- the use of rich tasks with multiple points of entry and more than one path to a solution,
- flexible grouping of students,
- multiple representations of mathematical concepts,
- writing in mathematics,
- monitoring of progress through on-going informal and formative assessments, and
- analysis of student work.

We hope that teachers will incorporate these strategies in each and every lesson.

It is hoped that you find this document useful as we strive to raise the mathematics achievement of all students in our district and state. Comments, questions, and suggestions for inclusions related to this document may be emailed to Dr. Dottie Whitlow, Executive Director, Mathematics and Science Department, Atlanta Public Schools, dwhitlow@atlantapublicschools.us.

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Explanation of the Terms and Categories Used in the Lesson Template

Task: This section gives the suggested number of days needed to teach the concepts addressed in a task, the task name, and the problem numbers of the task as listed in the Georgia Department of Education's Mathematics III Framework Teacher Edition (GaDOE TE).

In some cases new tasks or activities have been developed. These activities have been named by the writers.

Standard(s): Key standards addressed in the lesson are listed in this section. Standards listed first, in regular type, are from the Common Core State Standards for Mathematics. Standards in bold type are the corresponding standards from Mathematics III of the Georgia Performance Standards.

New Vocabulary: Vocabulary is listed here the *first* time it is used. It is strongly recommended that teachers, particularly those teaching Math Support, use interactive word walls. Vocabulary listed in this section should be included on the word walls and previewed in Math Support.

Mathematical concepts/skills: Major concepts addressed in the lesson are listed in this section whether they are CCGPS Math III concepts or were addressed in earlier grades or courses.

Prior knowledge: Prior knowledge includes only those topics studied in previous grades or courses. It does not include CCGPS Math III content taught in previous lessons.

Essential Question(s): Essential questions may be daily and/or unit questions.

Suggested materials: This is an attempt to list all materials that will be needed for the lesson, including consumable items, such as graph paper; and tools, such as graphing calculators and compasses. This list does not include those items that should always be present in a standards-based mathematics classroom such as markers, chart paper, and rulers.

Warm-up: A suggested warm-up is included with every lesson. Warm-ups should be brief and should focus student thinking on the concepts that are to be addressed in the lesson.

Opening: Openings should set the stage for the mathematics to be done during the work time. The amount of class time used for an opening will vary from day-to-day but this should not be the longest part of the lesson.

Worktime: The problem numbers have been listed and the work that students are to do during the worktime has been described. A student version of the day's activity follows the lesson template in every case. In order to address all of the standards in CCGPS Math III, some of the problems in some of the original GaDOE tasks have been omitted and less time consuming activities have been substituted for those problems. In many instances, in the student versions of the tasks, the writing of the original tasks has been simplified. In order to preserve all vocabulary, content, and meaning it is important that teachers work the original tasks as well as the student versions included here.

Teachers are expected to both facilitate and provide some direct instruction, when necessary, during the work time. Suggestions related to student misconceptions, difficult concepts, and deeper meanings have been included in this section.

Questioning is extremely important in every part of a standards-based lesson. We included suggestions for questions in some cases but did not focus on providing good questions as extensively as we would have liked. Developing good questions related to a specific lesson should be a focus of collaborative planning time.

Closing: The closing may be the most important part of the lesson. This is where the mathematics is formalized. Even when a lesson must be continued to the next day, teachers should stop, leaving enough time to "close", summarizing and formalizing what students have done to that point. As much as possible students should assist in presenting the mathematics learned in the lesson. The teacher notes are all important in determining what mathematics should be included in the closing.

Homework: In some cases, homework suggestions are provided. Teachers should use their resources, including the textbook, to assign homework that addresses the needs of their students.

Homework should be focused on the skills and concepts presented in class, relatively short (30 to 45 minutes), and include a balance of skills and thought-provoking problems.

Differentiated support/enrichment: On the topic of differentiation, it is critical to reiterate that many of the strategies used in a standards-based mathematics classroom promote differentiation. These strategies include

- the use of rich tasks with multiple points of entry and more than one path to a solution
- flexible grouping of students
- multiple representations of mathematical concepts
- writing in mathematics
- monitoring of progress through on-going informal and formative assessments
- and analysis of student work.

Check for understanding: A check for understanding is a short, focused assessment-a ticket out the door, for example. There are many good resources for these items on-line at www.georgiastandards.org, along with other GaDOE materials related to the standards.

Resources/materials for Math Support: Again, in some cases, we have provided materials and/or suggestions for Math Support. This section should be personalized to your students, class, and/or school, based on your resources.

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Common Core Georgia Performance Standards Mathematics III
Unit 3: Exponential and Logarithmic Functions

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Unit 1 Timeline

Task 1: How Long Does It Take?	3 days
Task 2: Finding the Inverses of Exponential Functions	4 days
Task 3: Modeling with Exponential and Logarithmic Functions	3 days

Task Notes

The importance of working the tasks used in these lessons cannot be overstated. In planning for the teaching of the Common Core Georgia Performance Standards in Mathematics, teachers should work the Student Tasks, read any corresponding teacher notes, and *then* examine the lessons provided here.

The tasks provided in this Supplement are based on the content of Unit 3 of the Georgia Department of Education's Mathematics III Framework. We suggest, as always, that teachers use this Supplement along with the GaDOE Teacher Edition which can be found on *Learning Village* on-line at www.georgiastandards.org.

Task 1: How Long Does It Take?
The concepts and skills addressed in this task include:

- writing exponential functions to model given situations
- solving exponential equations and inequalities algebraically, graphically, and numerically
- defining and understanding the properties of n^{th} roots
- extending properties of exponents to include rational exponents
- exploring real phenomena related to exponential functions including half-life and doubling time
- determining relationships between two functions that are inverses

This task combines parts of the first two tasks in the GaDOE Mathematics III Unit 3 Framework. Item numbers for *The Planet of Exponentia* and *How Long Does It Take* are noted at the beginning of each lesson. GaDOE Teacher Notes should be referenced for these items.

Task 2: Finding the Inverses of Exponential Functions
The concepts and skills addressed in this task include:

- defining logarithmic functions as inverses of exponential functions
- solving exponential and logarithmic equations analytically, graphically, and using appropriate technology
- exploring real phenomena related to exponential and logarithmic functions
- investigating and explaining characteristics of exponential and logarithmic functions including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, and rate of change
- graphing functions as transformations of $f(x) = a^x$ and $f(x) = \log_a x$

Items in this task are contained in the GaDOE task entitled *The Population of Exponentia*. Item numbers for *The Population of Exponentia* are noted at the beginning of each lesson. GaDOE Teacher Notes should be referenced for these items.

Task 3: Modeling with Exponential and Logarithmic Functions
The concepts and skills addressed in this task include:

- using exponential and logarithmic functions to model real-world situations
- solving problems related to real-world situations by solving exponential and logarithmic equations analytically and using appropriate technology
- investigating and explaining characteristics of exponential and logarithmic functions including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, and rate of change
- graphing functions as transformations of $f(x) = a^x$ and $f(x) = \log_a x$

Items in this task are contained in the GaDOE task entitled *Modeling Natural Phenomena on Earth.* Item numbers for *Modeling Natural Phenomena on Earth* are noted at the beginning of each lesson. GaDOE Teacher Notes should be referenced for these items.

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Task 1: How Long Does It Take?

CCGPS Mathematics III
Task 1: *How Long Does It Take?* **Day 1/3**
GaDOE *How Long Does It Take?* # 1
GaDOE *Planet of Exponentia* #3 and # 4
CCSS Standard(s):

Number and Quantity
Quantities N-Q
Reason quantitatively and use units to solve problems.

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

2. Define appropriate quantities for the purpose of descriptive modeling.

The Real Number System N-RN
Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define $5^{1/3}$ to be the cube root of 5 because we want $(5^{1/3})^3 = 5^{(1/3)3}$ to hold, so $(5^{1/3})^3$ must equal 5.*

2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Algebra
Seeing Structure in Expressions A-SSE
Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.*

   a. Interpret parts of an expression, such as terms, factors, and coefficients.

   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret $P(1+r)^n$ as the product of P and a factor not depending on P.*

Write expressions in equivalent forms to solve problems

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.*

   c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression 1.15^t can be rewritten as $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

Creating Equations* A-CED
Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable and use them to solve problems.

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

Represent and solve equations and inequalities graphically

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions.*

Functions

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.**

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.**

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*

e. Graph exponential and logarithmic functions, showing intercepts and end behavior.

Building Functions F-BF

Build a function that models a relationship between two quantities

1. Write a function that describes a relationship between two quantities.*

a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

Build new functions from existing functions

4. Find inverse functions.

b. (+) Verify by composition that one function is the inverse of another.

c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

Linear, Quadratic, and Exponential Models* F –LE

Interpret expressions for functions in terms of the situation they model

5. Interpret the parameters in a linear or exponential function in terms of a context.

GPS Standard(s):

MM3A2. Students will explore logarithmic functions as inverses of exponential functions.

a. Define and understand the properties of n^{th} roots.

b. Extend properties of exponents to include rational exponents.

e. Investigate and explain characteristics of exponential and logarithmic functions including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, and rate of change.

f. Graph functions as transformations of $f(x) = a^x$, $f(x) = \log_a x$, $f(x) = e^x$, $f(x) = \ln x$.

g. Explore real phenomena related to exponential and logarithmic functions including half-life and doubling time.

MM3A3. Students will solve a variety of equations and inequalities.

b. Solve exponential…equations analytically, graphically, and using appropriate technology.

c. Solve exponential…inequalities analytically, graphically, and using appropriate technology. Represent solution sets of inequalities using interval notation.

d. Solve a variety of types of equations by appropriate means choosing among mental calculation, pencil and paper, or appropriate technology.

New vocabulary: n^{th} roots

Mathematical concepts/skills:

- writing exponential functions (with integer exponents only) to model given situations
- solving exponential equations and inequalities graphically and numerically
- writing n^{th} roots as exponential expressions
- understanding $f(x) = x^n$ and $g(x) = x^{\frac{1}{n}}$ as functions that are inverses of each other when domains are appropriate
- determining characteristics of functions that are inverses

Prior knowledge:

- exponential functions with integer exponents as models of natural phenomena
- solving simple exponential equations and inequalities algebraically, graphically, and numerically
- properties of inverse functions

Essential question(s): How can I use exponential functions to solve real-world problems?

Suggested materials:

- graphing calculators
- graph paper

Warm-up: Post the following:

List three things you know about exponential functions.

Opening: Discuss student responses to the *Warm-up.* Students may remember some of the characteristics of exponential functions listed below. It is not necessary, in the opening discussion, to go beyond what students remember. Opportunities for reviewing characteristics of exponential functions will present themselves throughout the task.

Characteristics of exponential functions include:

- The standard form of an exponential function is $f(x) = ab^x$, where a can be any real number other than 0 and b can be any positive real number other than 1.
- Exponential functions are strictly increasing or strictly decreasing depending upon whether b is greater than 1 or less than 1.
- Graphs of exponential functions have horizontal asymptotes.
- Exponential functions grow at rate greater than linear or quadratic functions.
- Geometric sequences can be written as exponential functions with whole number domains.

Worktime: Students should work in pairs to complete *Items 1 and 2* of the task.

The material in *Item 1* should prompt a review of the concepts related to exponential functions learned in Math II. Have a class discussion of *Items 1a - e* before allowing students to begin work on *part f.*

In *Item 1f* students should write an inequality that can be used to determine when the amount of drug in the bloodstream will be less than 10 mg. At this point, they are asked to solve the inequality and the equation (required in *part g)* using a graphing calculator. Discuss various methods of finding a solution for the equation, including using the table feature and graphing Y_1 = (the function determined in *part d*), $Y_2 = 10$ and then using the *CALC, intersect* feature of the calculator. Solving the equation algebraically requires that students take the log of both sides of the equation which will be discussed in the next task.

The discussion preceding *Item 2* introduces n^{th} roots. Make sure students understand the concept of an n^{th} root before moving on to Item 2.

Closing: Allow students to share their responses to *Item 2*.

The purposes of *Item 2* include reinforcing the idea of an n^{th} root and reviewing the concepts related to functions that are inverses. It is important, for work in the next task, that students review the following:

- Only one-to-one functions (functions that pass the horizontal line test) have inverses that are also functions. Inverses of functions that are not one-to-one are inverse *relations.*
- To find the inverse of a function, we interchange x and y and solve for y.
- Domains and ranges of f and f^{-1} are interchanged.
- Two functions are inverses if and only if $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.
- Graphs of f and f^{-1} are reflections across the line $y = x$.

Homework: See attached.

Differentiated support/enrichment:

Check for understanding:

Resources/materials for Math Support: Students should preview:

- characteristics of exponential functions
- writing exponential functions with integer exponents to model given contexts
- solving simple exponential equations and inequalities algebraically, graphically, and numerically
- properties of exponents
- properties of inverse functions

CCGPS Mathematics III
Task 1: ***How Long Does It Take?***
Day 1 Student Task

As you learned in Math II, exponential functions are used to model a variety of natural phenomena. Medical personnel, particularly clinical pharmacists working in pharmacokinetics, use exponential functions to help them determine and monitor dosing regimens of potentially toxic drugs in critically ill patients. The idea is to get the level of drug in the bloodstream high enough to be effective but not so high that it becomes toxic.

1. Suppose for a specific patient and dosing regimen, a drug reaches a peak level of 300 milligrams. The drug is then eliminated from the bloodstream at a rate of about 20% per hour.

   a. How much of the drug remains 2 hours after the peak level? 5 hours after the peak level? Complete the table below showing how you obtained your answers.

Time since peak in hours	0	1	2	3	4	5
Vitamin concentration in bloodstream in milligrams	300					

   b. After 2 hours, the amount of drug left in the patient's bloodstream can be represented by the expression 300(1 – 0.2)(1 – 0.2). Explain why.

   c. Represent each value in the table above using an expression similar to the one in *part b*.

   d. Write a function that gives the drug level in the patient's bloodstream t hours after the peak level.

   e. Use the function you wrote in *part d* to compute values for the table in *part a*. Did you get the same results?

   f. After how many hours will there be less than 10 mg of the drug remaining in the bloodstream? Explain how you would determine this answer using both a graph and the table feature of your graphing calculator.

   g. Write an equation that you could solve to determine when the drug concentration is exactly 10 mg. Use you graphing calculator to help you solve the equation. Explain your method for solving.

To solve the equation written in *part g* algebraically, we must know how to find inverses of exponential functions. This topic will be explored later in this unit.

Note that if we could use only integer exponents for *t* in the function discussed in *Item 1*, our graph would be a set of discrete points rather than a smooth, continuous curve. It makes sense, in thinking about time, that our function be continuous. In order for this to happen, the domain of the function, values for the exponent *t*, must be all real numbers on a given interval, not just integer values. This raises the idea of rational exponents, that is, computing values such as $3^{\frac{3}{4}}$ or $\left(\frac{1}{2}\right)^{\frac{7}{3}}$. Exponents may be irrational numbers as well, although we will not deal with irrational exponents directly in this unit. It is enough to know that for an exponential function to be continuous on a given interval, the domain must be all real numbers on that interval.

We will begin our discussion of **rational exponents** by defining what is meant by an ***n*th root**.

You know already that a *square root* of *b* is a number whose square is *b*. The cube root of *b* is the number whose cube is *b*. Likewise, the n^{th} root of *b* is the number that when raised to the n^{th} power is *b*. For example, the 5^{th} root of 32 is 2 because $2^5 = 32$. We can write the 5^{th} root of 32 as $\sqrt[5]{32}$.

Another notation used to represent taking roots employs exponents. Instead of writing the 5^{th} root of 32 as $\sqrt[5]{32}$, we can write it as $32^{\frac{1}{5}}$. The cube root of 27 can be written as $27^{\frac{1}{3}}$. How do you think we would represent the n^{th} *root* of a number *x?*

2. Consider $f(x) = x^3$ and $g(x) = x^{\frac{1}{3}}$.

   a. How do you think the graphs of $f(x)$ and $g(x)$ are related? Test your conjecture by graphing both functions on the same coordinate plane.

   b. Considering that $x^{\frac{1}{3}} = \sqrt[3]{x}$, evaluate $f(g(x))$ and $g(f(x))$. What do the results tell you about $f(x)$ and $g(x)$?

   c. Using the results of *parts a* and *b* and what you remember from Math II, write a paragraph summarizing characteristics of inverses of functions, how to find inverses algebraically and graphically, and how to tell if inverses are functions.

CCGPS Mathematics III
Task 1: *How Long Does It Take?*
Day 1 Homework

1. A car that originally cost \$ 56,000 depreciates at a rate of 15% per year.
    a. Write a function that represents the value of the car after t years.
    b. How much is the car worth after 8 years?

2. Stock in a Fortune 500 company has increased in value at an average rate of 12% per year for the past 5 years.
    a. Write a function that will represent the value of the stock in t years given that the rate of increase remains the same and the initial value of the stock is represented by A_0.
    b. If your aunt gives you \$2000 to invest in this stock, how much will it be worth in 10 years?

3. Consider the function $f(x) = x^2 + 3$.
    a. $f(x)$ does not have an inverse that is also a function. Explain why.
    b. Restrict the domain of f to an interval over which the function is one-to-one. Call the new function $g(x)$. State both the domain and the range of $g(x)$.
    c. Find $g^{-1}(x)$ and state both the domain and range.
    d. Verify that $g(x)$ and the function you found in *part c* are inverses.
    e. Graph $g(x)$ and $g^{-1}(x)$ on the same coordinate plane and describe the relationship between the two graphs.

4. The formula $S = 4\pi r^2$ gives the surface area of a sphere in terms of its radius.
    a. Rewrite the formula so that it gives the radius as a function of the surface area.
    b. Find, to the nearest tenth, the radius of a sphere with surface area of 25 in^2.

5. A sphere has a volume of 233 in^3.
    a. Write a formula that gives the radius of the sphere as a function of its volume.
    b. Find the radius of the sphere to the nearest tenth of an inch.

6. Express each radical expression in exponential form and evaluate.
    a. $\sqrt[5]{243}$
    b. $\sqrt[4]{256}$
    c. $\sqrt[3]{-343}$
    d. $\sqrt{289}$

CCGPS Mathematics III
Task 1: ***How Long Does It Take?*** **Day 2/3**
(GaDOE #'s 2 and 3)

CCSS Standard(s):

Number and Quantity
Quantities N-Q
Reason quantitatively and use units to solve problems.

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

2. Define appropriate quantities for the purpose of descriptive modeling.

The Real Number System N-RN
Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define* $5^{1/3}$ *to be the cube root of 5 because we want* $(5^{1/3})^3 = 5^{(1/3)3}$ *to hold, so* $(5^{1/3})^3$ *must equal 5.*

2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Algebra
Seeing Structure in Expressions A-SSE
Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.*

   a. Interpret parts of an expression, such as terms, factors, and coefficients.

   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of P and a factor not depending on P.*

Write expressions in equivalent forms to solve problems

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.*

   c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

Creating Equations* A-CED
Create equations that describe numbers or relationships

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

Represent and solve equations and inequalities graphically

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

Functions

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.**

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.**

6. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.*

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*

   e. Graph exponential and logarithmic functions, showing intercepts and end behavior.

8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

   b. Use the properties of exponents to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, and classify them as representing exponential growth or decay.

9. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbaldescriptions). *For example, given a graph of one quadratic function and an algebraic expression for another, say which has the larger maximum.*

Building Functions F-BF

Build a function that models a relationship between two quantities

1. Write a function that describes a relationship between two quantities.*

   a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

Linear, Quadratic, and Exponential Models* **F –LE**

Construct and compare linear, quadratic, and exponential models and solve problems.

1. Distinguish between situations that can be modeled with linear functions and with exponential functions.

   c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

Interpret expressions for functions in terms of the situation they model

5. Interpret the parameters in a linear or exponential function in terms of a context.

GPS Standard(s):

MM3A2. Students will explore logarithmic functions as inverses of exponential functions.

a. Define and understand the properties of n^{th} roots.
b. Extend properties of exponents to include rational exponents.
e. Investigate and explain characteristics of exponential and logarithmic functions including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, and rate of change.
f. Graph functions as transformations of $f(x) = a^x$, $f(x) = \log_a x$, $f(x) = e^x$, $f(x) = \ln x$.
g. Explore real phenomena related to exponential and logarithmic functions including half-life and doubling time.

New vocabulary: rational exponents

Mathematical concepts/skills:

- reviewing relationships among sets of numbers, including real numbers, integers, complex numbers, irrational numbers, rational numbers, imaginary numbers, whole numbers, and counting numbers
- writing n^{th} roots as exponential expressions
- simplifying algebraic expressions using the laws of exponents
- writing exponential functions to model given situations
- using exponential functions as models to solve real-world problems
- comparing graphs of exponential functions
- calculating an average rate of change over a given interval

Prior knowledge:

- relationships among sets of numbers, including real numbers, integers, complex numbers, irrational numbers, rational numbers, imaginary numbers, whole numbers, and counting numbers
- simplifying algebraic expressions (integral exponents only) using the laws of exponents
- exponential functions with integer exponents as models of natural phenomena
- using exponential functions as models to solve real-world problems
- comparing graphs of exponential functions
- calculating an average rate of change over a given interval

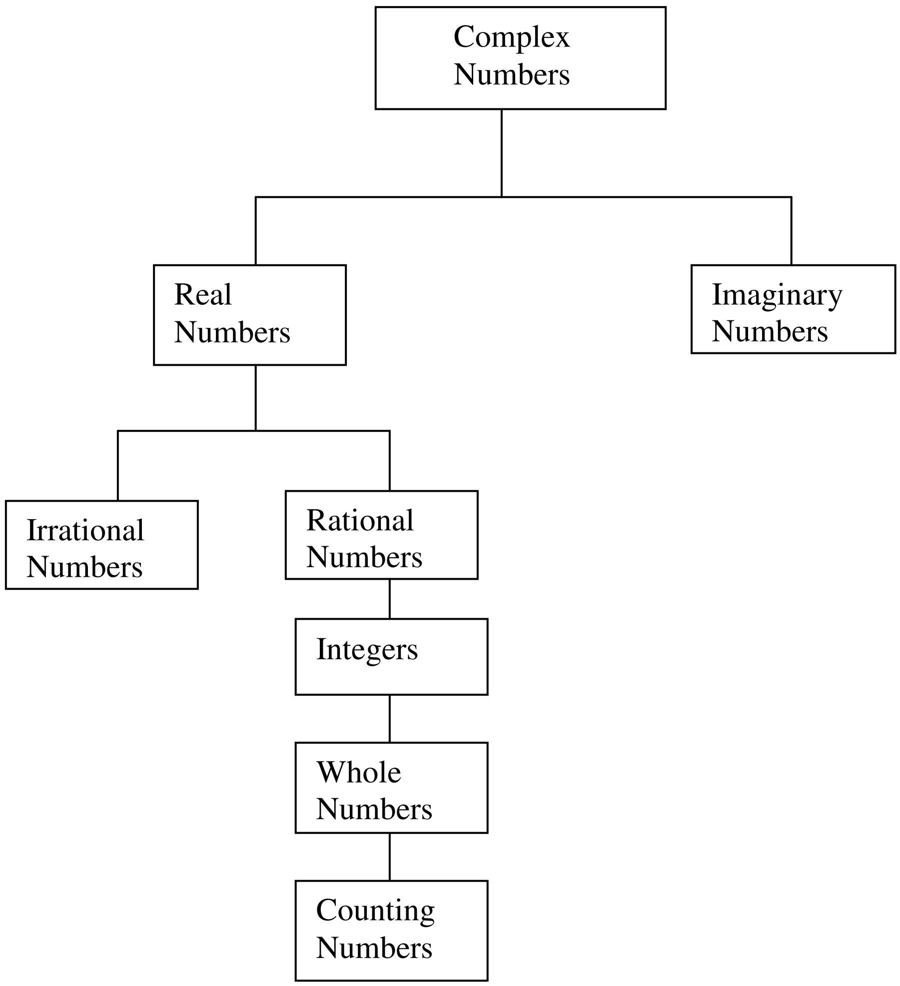
Essential question(s): How can I use exponential functions to solve real-world problems?

Suggested materials:
- graphing calculators
- graph paper

Warm-up: Post the following:

> *In previous courses, you studied different sets of numbers, including real numbers, integers, complex numbers, irrational numbers, rational numbers, imaginary numbers, whole numbers, and counting numbers.*
>
> *Draw a diagram that shows the relationships among these sets of numbers.*

Opening: Discuss student responses to the *Warm-up.* This is *Item 3a* of the task. It is very important at this point for students to understand the relationships among the sets of numbers named. A diagram similar to the one below or Venn diagrams are appropriate. Questions that might be asked include:
- Explain why the set of real numbers is a subset of the set of complex numbers.
- Give an example of a complex number that is a real number.
- Give an example of a complex number that is not a real number.
- What are imaginary numbers?
- What is the definition of a rational number?
- What is an irrational number?
- Give an example of an irrational number written in radical form.
- Give an example of an irrational number written in decimal form.
- How is the set of integers different from the set of rational numbers?
- How is the set of counting numbers different from the set of whole numbers?



Worktime: Students should work in pairs to complete *Items 3b – 5* of the task.

After students have had sufficient time to complete the *Laws of Exponents* and *Item 4a*, discuss these items and the definition of rational exponents before allowing students to begin *Item 4b*.

Have another short discussion of *Items 4b – d* before allowing students to begin *Item 5*.

Monitor work carefully as students complete *Items 5 a – e*. Allow students to share their responses to these items before beginning *Item 5f*.

Closing: Allow students to share their responses to *Items 5f – j*. In comparing the two graphs in *part i,* the following points should be addressed:

- Both graphs have domains [0. ∞) and ranges (0, 80].
- Both graphs are strictly decreasing.
- Both have horizontal asymptotes $y = 0$.
- Both graphs have y- intercepts of 80.
- If $f(x) = 80(0.5)^{\left(\frac{t}{5}\right)}$ and $g(x) = 80(0.5)^{\left(\frac{t}{3}\right)}$, then $g(x)$ is decreasing faster than $f(x)$.

(Note: It will be particularly helpful to have students post these graphs on Anchor Charts as they will need to refer to them again in the next lesson.

The average rate of change for each of the two functions, over the interval from 1 to 2 hours, is calculated as follows:

$$average\ rate\ of\ change = \frac{f(2) - f(1)}{2-1} = \frac{60.629 - 69.644}{1} = -9.016$$

$$average\ rate\ of\ change = \frac{g(2) - g(1)}{2-1} = \frac{50.397 - 63.496}{1} = -13.099$$

This tells us that on the interval from 1 to 2 hours, $g(x)$ is decreasing at an average rate of approximately 13 milligrams per hour, while $f(x)$ is decreasing at an average rate of approximately 9 milligrams per hour.

(Note: The ability to calculate an average rate of change is a necessary skill for the GHSGT.)

Homework: See attached.

Differentiated support/enrichment:

Check for understanding:

Resources/materials for Math Support: Students should preview:

- relationships among sets of numbers, including real numbers, integers, complex numbers, irrational numbers, rational numbers, imaginary numbers, whole numbers, and counting numbers
- properties of exponents
- characteristics of graphs of exponential functions
- calculating average rate of change over a given interval

CCGPS Mathematics III
Task 1: *How Long Does It Take?*
Day 2 Student Task

3. **Rational Exponents.** In previous courses, you studied different sets of numbers, including real numbers, integers, complex numbers, irrational numbers, rational numbers, imaginary numbers, whole numbers, and counting numbers.

   a. Draw a diagram that shows the relationships among these sets of numbers.

   b. What is the difference between integer exponents and rational exponents?

4. **The laws for integer exponents apply to all real number exponents.** As a means of review, complete the following laws of exponents. (If you don't remember the laws from your previous classes, try some examples to help you.)

   For all real numbers *m*, *n*, *a*, and *b*, where $a > 0$ and $b > 0$,

   $a^0 =$ ____________ $a^1 =$ ____________ $a^n =$ ____________

   $(a^m)(a^n) =$ ________ $\dfrac{(a^m)}{(a^n)} =$ ________ $a^{-n} =$ ____________

   $(a^m)^n =$ __________ $(ab)^m =$ ________ $\left(\dfrac{a}{b}\right)^m =$ __________

   If $a^m = a^n$, *then m* ___ *n.*

   a. You have just learned that the *n*th root of a number *x* can be represented as $x^{\frac{1}{n}}$.

      i. Using your laws of exponents, write another expression for $\left(x^{\frac{1}{n}}\right)^m$.

      ii. Using your laws of exponents, write another expression for $\left(x^m\right)^{\frac{1}{n}}$.

      iii. What do you notice about the answers in (i) and (ii)? What does this tell you about rational exponents?

This leads us to the definition of **rational exponents**.

For $a > 0$, and integers m and n, with $n > 0$,

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(a^m\right)^{\frac{1}{n}}$$

b. Rewrite the following using positive rational exponents in simplest form.

i. $\sqrt[7]{x^3}$ ii. $\left(\frac{1}{x}\right)^{-5}$ iii. $\left(\sqrt{x}\right)^6$ iv. $\frac{1}{\sqrt[3]{x^5}}$

c. Simplify each of the following. Show your steps. For example, $27^{\frac{2}{3}} = \left(27^{\frac{1}{3}}\right)^2$ or $3^2 = 9$.

i. $16^{\frac{3}{4}}$ ii. $36^{\frac{3}{2}}$ iii. $81^{\frac{7}{4}}$

To solve equations such as $x^3 = 27$, we take the cube root of both sides or we can raise both sides of the equation to the $\frac{1}{3}$ power (the reciprocal of the exponent).

To solve $x^{\frac{3}{2}} = 27$:

- we can square both sides and then take the cube root,
- we can take the cube root of both sides and then square both sides, or
- we can raise both sides to the $\frac{2}{3}$ power.

$$x^{\frac{3}{2}} = 27 \rightarrow \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} = 27^{\frac{2}{3}} \rightarrow x = \left(27^{\frac{1}{3}}\right)^2 \rightarrow x = 3^2 = 9$$

d. Rewrite each of the following using rational exponents and use inverses to solve for the variable. (You may need to use a calculator for some of them. Be careful!)

i. $\sqrt[5]{b} = 2$ ii. $\sqrt[5]{c^3} = 4.2$ iii. $\frac{1}{\sqrt[4]{d}} = \frac{1}{5}$

5. A can of Instant Energy, a 16-ounce energy drink, contains 80 mg of caffeine. Suppose the caffeine in the bloodstream peaks at 80 mg. The half-life of caffeine is 5 hours. In other words, ½ of the caffeine has been eliminated from the bloodstream after 5 hours.

   a. How much caffeine remains in the bloodstream after 5 hours? 10 hours? 1 hour? 2 hours? Make a table to organize your answers. Explain how you came up with your answers. (You can return to your answers later to make any corrections, if your method was incorrect.)

Time since peak in hours	0	1	2	5	10
Caffeine in the bloodstream in milligrams	80				

   b. In *Item 1*, 80% of the existing drug remained in the bloodstream after each hour. Fifty percent of the caffeine from Instant Energy remains in the bloodstream after each **5** hours.

      i. In *Item 1*, what did the exponent in your function represent?

      ii. For a function representing the amount of caffeine remaining in the bloodstream, the exponent needs to represent the number of 5-hour time periods that have elapsed. If you represent 1 hour as $\frac{1}{5}$ of a 5-hour time period, how do you represent 2 hours? 3 hours? 10 hours? t hours?

   c. Using your last answer in *part b* as your exponent, write an exponential function to model the amount of caffeine remaining in the blood stream t hours after the peak level.

   d. Use the function you wrote in *part c* to check your answers for the table in *part a*. Make any necessary corrections. (Be careful when entering rational exponents in the calculator. Use parentheses.)

   e. Determine the amount of caffeine remaining in the bloodstream 3 hours after the peak level. What about 8 hours after peak level? 20 hours?

   f. Suppose the half-life of caffeine in the bloodstream was 3 hours instead of 5.

      i. Write a function representing this new half-life.

      ii. Determine the amount of caffeine in the bloodstream after 1 hour, 2 hours, 5 hours, and 10 hours. (You need to consider how many 3-hour time intervals are used in each time value.)

      iii. Which half-life results in the caffeine being eliminated faster? Explain why this makes sense.

g. Problems involving roots and rational exponents can sometimes be solved by rewriting expressions in a different form.

   i. Consider again the function in *part a* resulting from a half-life of caffeine of 5 hours. Use the laws of exponents to rewrite the function in order to determine the percent of caffeine remaining in the bloodstream *each hour*. What percent of caffeine remains in the bloodstream each hour?

   ii. Suppose the half-life of caffeine in an adult's bloodstream is 3 hours. What percent of caffeine remains in the bloodstream *each hour*?

h. Graph the functions from *parts c* and *f* on graph paper on the same coordinate plane.

i. Compare the graphs of the two functions. How are the graphs similar? Different? What are the intercepts? When are they increasing/decreasing? Are there asymptotes? If so, what are they? Do the graphs intersect? Where?

j. Calculate the average rate of change for each of the two functions, over the interval from 1 to 2 hours. Explain what the average rates of change tell us in this situation.

CCGPS Mathematics III
Task 1: *How Long Does It Take?*
Day 2 Homework

Evaluate each expression.

1. $(-64)^{\frac{2}{3}}$

2. $25^{\frac{3}{2}}$

3. $\sqrt[4]{625^2}$

Simplify each algebraic expression. Write final answers with nonnegative exponents.

4. $\left(2m^{\frac{1}{2}}n^{\frac{1}{3}}\right)\left(3m^2n^{\frac{5}{3}}\right)$

5. $\dfrac{3x^{\frac{1}{2}}}{6x^{\frac{-2}{3}}}$

6. $\left(3x^{\frac{-1}{2}}y^{\frac{3}{4}}\right)^4$

7. The population of a certain city triples every 20 years.
   a. If A_0 represents the population of the city on January 1 of the current year, write a function to model the population in *t* years.

   b. Use the laws of exponents to determine the average yearly growth rate of the population to the nearest tenth of a percent.

   c. If the population continues to grow at approximately the same rate and there were 2 million people living in the city on January 1, 2010, how many people will be living there on January 1, 2033?

CCGPS Mathematics III
Task 1: ***How Long Does It Take?*** **Day 3/3**
(GaDOE # 4)

CCSS Standard(s):

Number and Quantity
Quantities N-Q
Reason quantitatively and use units to solve problems.

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

2. Define appropriate quantities for the purpose of descriptive modeling.

The Real Number System N-RN
Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define* $5^{1/3}$ *to be the cube root of 5 because we want* $(5^{1/3})^3 = 5^{(1/3)3}$ *to hold, so* $(5^{1/3})^3$ *must equal 5.*

2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Algebra
Seeing Structure in Expressions A-SSE
Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.*

   a. Interpret parts of an expression, such as terms, factors, and coefficients.

   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of P and a factor not depending on P.*

Write expressions in equivalent forms to solve problems

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.*

   c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

Creating Equations* A-CED
Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable and use them to solve problems.

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

Represent and solve equations and inequalities graphically

10. Understand that the graph of an equation in two variables is the set of all its solutions plotted in the coordinate plane, often forming a curve (which could be a line).

11. Explain why the x-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, absolute value, exponential, and logarithmic functions

Functions

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.**

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.**

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*

   e. Graph exponential and logarithmic functions, showing intercepts and end behavior.

8. Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

   b. Use the properties of exponents to interpret expressions for exponential functions. For example, identify percent rate of change in functions such as $y = (1.02)^t$, $y = (0.97)^t$, $y = (1.01)^{12t}$, $y = (1.2)^{t/10}$, and classify them as representing exponential growth or decay.

Building Functions F-BF

Build a function that models a relationship between two quantities

1. Write a function that describes a relationship between two quantities.*

   a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

Linear, Quadratic, and Exponential Models* **F –LE**

Construct and compare linear, quadratic, and exponential models and solve problems.

1. Distinguish between situations that can be modeled with linear functions and with exponential functions.

   c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

Interpret expressions for functions in terms of the situation they model

5. Interpret the parameters in a linear or exponential function in terms of a context.

GPS Standard(s):

MM3A2. Students will explore logarithmic functions as inverses of exponential functions.

a. Define and understand the properties of n^{th} roots.
b. Extend properties of exponents to include rational exponents.
e. Investigate and explain characteristics of exponential and logarithmic functions including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, and rate of change.
f. Graph functions as transformations of $f(x) = a^x$, $f(x) = \log_a x$, $f(x) = e^x$, $f(x) = \ln x$.
g. Explore real phenomena related to exponential and logarithmic functions including half-life and doubling time.

MM3A3. Students will solve a variety of equations and inequalities.

b. Solve exponential…equations analytically, graphically, and using appropriate technology.
c. Solve exponential… inequalities analytically, graphically, and using appropriate technology. Represent solution sets of inequalities using interval notation.
d. Solve a variety of types of equations by appropriate means choosing among mental calculation, pencil and paper, or appropriate technology.

New vocabulary: hourly growth rate

Mathematical concepts/skills:

- writing exponential functions to model given situations
- using exponential functions as models to solve real-world problems
- comparing graphs of exponential functions
- using properties of exponents to determine percent rates of growth
- solving radical and exponential equations algebraically and graphically

Prior knowledge:

- writing exponential functions to model given situations
- using exponential functions as models to solve real-world problems
- comparing graphs of exponential functions
- using laws of exponents
- solving radical and exponential equations algebraically and graphically

Essential question(s): How can I use exponential functions to solve real-world problems?

Suggested materials:

- graphing calculators
- graph paper

Warm-up: Post the following:

> *Working with your partner, compare responses to homework. Be prepared to share your solutions and to ask any remaining questions you have related to the assignment.*

Opening: Discuss student responses to the homework. *Problem 7 b* is particularly important and will set the stage for today's lesson. When discussing elimination of a drug from the bloodstream, students were asked to rewrite functions in order to determine the amount of drug *remaining in the bloodstream after each hour*. The amount of the drug in the body was decreasing. The percent of drug *remaining* in the body was $100(1 - r)\%$ where $100r$ was the percent decrease each time period.

The population of the city discussed in *Problem 7* of the homework is growing or *increasing* and we would like to know the *yearly growth rate*. In *parts a* and *b* students find that the function $f(t) = A_0(3)^{\frac{t}{20}} = A_0\left(3^{\frac{1}{20}}\right)^t \approx A_0(1.0564)^t$. It is important for students to understand that multiplying the existing population by *1.0564* is the same as multiplying by $(1 + .0564)$. At the end of the first year, for example, we have $A_0(1 + .0564) = A_0(1) + A_0(.0564)$ or, in other words, the existing population plus approximately 5.64% of the existing population. Each year the population grows by a constant factor of approximately 5.64%. In general, if a quantity is growing (or decreasing) exponentially, it is growing (or decreasing) by a **constant factor** as opposed to a constant amount as is the case with linear functions. If the general function is $g(t) = A_0(1 + r)^t$ where A_0 is the initial amount, the population is growing at a constant factor of r or $100r$ percent.

Worktime: Students should work in pairs to complete *Item 6* of the task.

After sufficient time has been given to complete *Item 6h*, allow students to share their work. (The graphs referred to in *Item 6h* should have been posted as Anchor Charts during the Closing of the previous lesson). Characteristics of the graphs in *Items 6g* and *6h* should be discussed thoroughly.
(Note: It may be beneficial to post the graphs in *Item 6g* along with those already posted so that all four graphs may be compared with those to be drawn in the next task.)

Allow time for students to complete *Items i – l* and then have another short discussion of these items.

Closing: Allow students to share their responses to *Items 6m – o.*

Homework:

Differentiated support/enrichment:

Check for understanding:

Resources/materials for Math Support: Students should preview:

- comparing graphs of exponential functions
- using laws of exponents
- solving radical and simple exponential equations algebraically and graphically

CCGPS Mathematics III
Task 1: *How Long Does It Take?*
Day 3 Student Task

Bacteria growth is another situation that can be modeled using exponential functions.

6. Suppose a culture begins with 25 bacteria.
    a. If the number of bacteria doubles each hour, how many bacteria are alive after 1 hour? 2 hours?

    b. Complete the chart below.

Time (hours)	0	1	2	3	4	5	6
Population	25	50					

    c. Write a function that represents the population of bacteria after *t* hours. (Check that your function gives you the same answers you determined above.

    d. How many bacteria are present after 7 ½ hours? After 15 hours?

    e. Suppose the initial bacteria population was 60 instead of 25. Write a function that represents the population of bacteria after *t* hours. Find the population after 7 ½ hours and 15 hours.

    f. Graph the functions in part (c) and (e).

    g. Compare the graphs in *part f*. Your discussion should include intercepts, intervals of increase/decrease, asymptotes, and points of intersection.

    h. Revisit the graphs in *Item 5h*. Compare with the graphs above. How are they similar and different? How can you tell by looking at the functions whether their graphs will be increasing or decreasing?

    i. A bacteria colony begins with 25 bacteria and doubles every 4 hours. Write a function for the number of bacteria present after *t* hours.

    j. The population of the city discussed in *Problem 7* of your homework assignment was growing or *increasing* and we asked you to find the *yearly growth rate*. You found that $f(t) = A_0(3)^{\frac{t}{20}} = A_0\left(3^{\frac{1}{20}}\right)^t \approx A_0(1.0564)^t$. It is important to understand that multiplying the existing population by *1.0564* is the same as multiplying by *(1 + .0564)*. At the end of the first year, for example, we have $A_0(1 + .0564) = A_0(1) + A_0(.0564)$ or, in other words, the existing population plus approximately 5.64% of the existing population. Each year the population grows by a constant factor of approximately 5.64%. In general form, if a quantity is growing (or decreasing) exponentially, it is growing (or decreasing) by a **constant factor** as opposed to a constant amount as is the case with linear functions. If the general function is $g(t) = A_0 (1 + r)^t$ where A_0 is the initial amount, the population is growing at a constant factor of r or $100r$ percent. If the general function is $g(t) = A_0 (1 - r)^t$ where A_0 is the initial amount, the population is decreasing at a constant factor of r or $100r$ percent.

What is the hourly growth rate of the bacteria colony in *part i*? Show how you know.

k. If there are originally 25 bacteria, at what rate is the colony growing per hour if the population of the bacteria doubles in 5 hours?

l. Another way to determine the growth rate in *part j* is to solve the equation $50 = 25(1 + r)^5$. Explain why.

m. What is the hourly growth rate of the bacteria if the population triples every 5 hours?

n. If there are originally 25 bacteria and the population doubles each hour, how long will it take the population to reach 100 bacteria? Solve the problem algebraically.

o. If there are originally 60 bacteria and the population doubles each hour, how long will it take the population to reach 100 bacteria? Explain how you solved the problem. (Solving the problem algebraically will be addressed later in the unit.)

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Task 2: Exponential Functions and Their Inverses

CCGPS Mathematics III
Task 2: ***Exponential Functions and Their Inverses*** **Day 1/4**
GaDOE *The Population of Exponentia # 2 a, b,* and *c*
CCSS Standard(s):

The Real Number System N-RN
Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define* $5^{1/3}$ *to be the cube root of 5 because we want* $(5^{1/3})^3 = 5^{(1/3)3}$ *to hold, so* $(5^{1/3})^3$ *must equal 5.*

2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Functions
Interpreting Functions F-IF
Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*

   e. Graph exponential and logarithmic functions, showing intercepts and end behavior.

Building Functions F-BF
Build new functions from existing functions

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

GPS Standard(s):
MM3A2. Students will explore logarithmic functions as inverses of exponential functions.

a. Define and understand the properties of n^{th} roots.
b. Extend properties of exponents to include rational exponents.
e. Investigate and explain characteristics of exponential and logarithmic functions including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, and rate of change.
f. Graph functions as transformations of $f(x) = a^x$, $f(x) = \log_a x$, $f(x) = e^x$, $f(x) = \ln x$.

New vocabulary:

Mathematical concepts/skills:

- investigating characteristics of exponential functions including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, and rate of change
- graphing functions as transformations of $f(x) = a^x$

Prior knowledge:

- investigating characteristics of exponential functions with integer exponents including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, and rate of change
- graphing functions as transformations of $f(x) = a^x$

Essential question(s): How can I determine the characteristics of an exponential function?

Suggested materials:

- graphing calculators
- graph paper

Warm-up: Post the following:

Given the function $y = \left(\frac{1}{2}\right)^x$ *, complete the table below by determining the y-value of the function for each value of x.*

Write $\left(\frac{1}{2}\right)^{-\frac{1}{2}}$ *using radical notation.*

x	y
-2	
-1	
$-\frac{1}{2}$	
0	
$\frac{1}{2}$	
1	
2	

Opening: Discuss the warm-up and allow time for students to graph the function. It is important for students to be able to apply the properties of exponents to evaluate expressions such as

$\left(\frac{1}{2}\right)^{-2} = 2^2 = 4$ and $\left(\frac{1}{2}\right)^{-\frac{1}{2}} = 2^{\frac{1}{2}} = \sqrt{2} \approx 1.41.$

Worktime: Students should work in pairs to complete *Items 1 - 4* of the task.

When students have had an opportunity to complete *Item 1 - 4*, have a class discussion of exponential functions and their graphs. An Anchor Chart containing characteristics of graphs of exponential functions for which $b > 1$ and $0 < b < 1$ would be beneficial. It is also important to note that exponential functions pass the *horizontal line test* meaning that they are *one-to-one functions* and thus have inverses that are also functions.

Closing: Allow students to share their responses to *Item 4*.

Homework:

Differentiated support/enrichment:

Check for understanding:

Resources/materials for Math Support: Students should preview:

- characteristics of exponential functions
- graphing transformations of exponential functions
- characteristics of inverse functions

CCGPS Mathematics III
Task 2: ***Exponential Functions and Their Inverses***
Day 1 Student Task

In Math II you learned that an exponential function is any function in the form $f(x) = ab^{x}$, where a is any real number other than 0, and b is any positive number other than 1.

In this task we will revisit graphs of exponential functions and then investigate the inverses of these functions both graphically and algebraically.

1. For each of the 5 functions $f(x) = b^x$, where $b = \frac{1}{2}, \frac{3}{4}$, 2, 5, and $\frac{3}{2}$:

   a. Complete the table below by finding a y-value for each value of x.

x	y
-2	
-1	
$-\frac{1}{2}$	
0	
$\frac{1}{2}$	
1	
2	

   b. Use the points determined in *part a* to graph the function on graph paper.

2. Compare the 5 graphs in *Item 1* by discussing each of the following:

   a. domain and range

   b. intercepts

   c. asymptotes

   d. intervals of increase and decrease

3. Discuss how the value of the base b affects an exponential function.

4. Consider your graph of $f(x) = 2^x$. Describe how each of the functions listed below would transform the graph of f. Check your conjecture by graphing the functions using your graphing calculator. The points (-2, ¼) and (2, 4) are on the graph of $f(x) = 2^x$. Give the location of these points after each transformation.

   a. $g(x) = 2^{x-3}$
      transformation ____
      (- 2, ¼) → ________________
      (2, 4) → ________________

   b. $j(x) = 3(2^x)$
      transformation ____
      (- 2, ¼) → ________________
      (2, 4) → ________________

   c. $k(x) = 2^{-x}$
      transformation ____
      (- 2, ¼) → ________________
      (2, 4) → ________________

   d. $l(x) = -2^x$
      transformation ____
      (- 2, ¼) → ________________
      (2, 4) → ________________

   e. $m(x) = 2^x + 3$
      transformation ____
      (- 2, ¼) → ________________
      (2, 4) → ________________

   f. $n(x) = 2^x - 1$
      transformation ____
      (- 2, ¼) → ________________
      (2, 4) → ________________

CCGPS Mathematics III
Task 2: ***Exponential Functions and Their Inverses*** **Day 2/4**
GaDOE *The Population of Exponentia # 2d, e,* and *f, #3c, d*

CCSS Standard(s):
Functions
Interpreting Functions F-IF
Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*

e. Graph exponential and logarithmic functions, showing intercepts and end behavior.

Building Functions F-BF
Build new functions from existing functions

4. Find inverse functions.

b. (+) Verify by composition that one function is the inverse of another.

c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

5. (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

GPS Standard(s):
MM3A2. Students will explore logarithmic functions as inverses of exponential functions.

c. Define logarithmic functions as inverses of exponential functions.
e. Investigate and explain characteristics of exponential and logarithmic functions including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, and rate of change.

MM3A3. Students will solve a variety of equations and inequalities.

b. Solve… logarithmic equations analytically, graphically, and using appropriate technology.
d. Solve a variety of types of equations by appropriate means choosing among mental calculation, pencil and paper, or appropriate technology.

New vocabulary: logarithmic function, logarithm of x to the base b

Mathematical concepts/skills:

- procedure for finding the inverse of a function
- characteristics of inverse functions
- graphing the inverse of an exponential function
- defining the logarithmic function
- converting from exponential to logarithmic and logarithmic to exponential functions
- evaluating logarithmic expressions
- solving logarithmic equations

Prior knowledge:
- procedure for finding the inverse of a function
- characteristics of inverse functions

Essential question(s): How do I define the inverse of an exponential function? How is this new function used to solve equations?

Suggested materials:
- graphing calculators
- graph paper

Warm-up: Post the following:

Find the inverse of f(x) = 3x – 4.
Describe the procedure you used for finding the inverse of this function.

Opening: Discuss the warm-up. Students should remember that to find the inverse of a function that is one-to-one, we interchange *x* and *y* and solve for *y*.

Worktime: Students should work in pairs to complete *Items 5 - 8* of the task.

In responding to *Item 5*, students should remember the following:
- Only one-to-one functions have inverses that are also functions.
- Domains and ranges of inverse functions are interchanged.
- $f(f^{-1}(x)) = f^{-1}(f(x)) = x$
- Graphs of inverse functions are reflections of each other over the line $y = x$.

These ideas should be stressed in preparation for the GHSGT.

After students have had an opportunity to complete *Item 6*, allow them to share characteristics of the graph of the inverse of $f(x) = 2^x$. Conduct a mini-lesson on the material contained in the task related to the definition of the logarithmic function.

Students should complete *Items 7*and *8.*

Closing: Allow students to share their responses to *Items 7* and *8.* Make sure that all students understand the definition of $f(x) = \log_b x$, including restrictions on the base and on the domain.

Homework: See attached.

Differentiated support/enrichment:

Check for understanding:

Resources/materials for Math Support: Students should preview:
- procedure for finding the inverse of a function
- characteristics of inverse functions

CCGPS Mathematics III
Task 2: *Exponential Functions and Their Inverses*
Day 2 Student Task

5. In the next few items we will define the inverse of an exponential function. As preparation, describe the procedure for finding the inverse of a function and then list the characteristics of two functions that are inverses of each other.

6. Carefully trace the coordinate plane and your graph of the function $f(x) = 2^x$ onto a piece of patty paper.

    a. Draw the line $y = x$ on your graph.

    b. Fold the paper along the line $y = x$.

    c. Trace the curve, as you see it, on the outside of the paper.

    d. Open the paper and trace the outside curve on the inside so that you have both the graph of the exponential function and the graph of its inverse on the front of the patty paper.

    e. Using the graph on your patty paper, describe the following characteristics of the inverse of the exponential function.

    Domain: _________________ Range: ____________________
    Asymptote: _______________ Intercept(s): __________________

The inverse of an exponential function is called a **logarithmic function.** In an exponential function, the input, or independent variable, is the exponent and the output, or dependent variable, is the value when the base is raised to that exponent. For example, in $f(x) = 3^x$, if $x = 4$, then $f(x) = 3^4 = 81$. Since the domains and ranges of inverse functions are interchanged, the independent variable in a logarithmic function is the value of a base raised to a given power and the dependent variable is the exponent on the base that gives that power.

To find the inverse of an exponential function, we use the same procedure we have used to find inverses of other functions- interchange x and y and then solve for y. Our first step is to start with $y = b^x$. Interchanging x and y we get the inverse $x = b^y$. In words, y is the power on b that gives x. The problem is that, to this point, you have had no notation that would allow you to express y symbolically. We will now introduce that notation. y is referred to as the "logarithim of x to the base b" and we solve $x = b^y$ for y by writing $y = \log_b x$. The formal definition is stated as follows:

$$y = \log_b x \text{ if and only if } b^y = x$$

Note that the "base" is the "base" in both expressions.

Look at a few examples:

- $10^2 = 100$ can be written as $\log_{10} 100 = 2$. Notice that the logarithm is the power on 10 that gives 100.
- $\log_4 64 = 3$ because $4^3 = 64$.
- Suppose $\log_x 144 = 2$. This is the same as saying $x^2 = 144$. Given that the base must be positive, the value of x is 12.

7. The following problems will give you some practice with logarithms. You should be able to work all parts of this item without a graphing calculator.

   a. Write each exponential equation as a logarithmic equation and each logarithmic equation as an exponential equation.

      i. $\log_{10}\left(\frac{1}{100}\right) = -2$

      ii. $5^3 = 125$

      iii. $\log_2 32 = 5$

   b. Evaluate each logarithm.

      i. $\log_{10}(0.1)$

      ii. $\log_3 81$

      iii. $\log_2\left(\frac{1}{16}\right)$

      iv. $\log_9 81$

   c. Write each logarithmic equation as an exponential equation and then solve for x.

      i. $\log_x 36 = 2$

      ii. $\log_4 4 = x$

      iii. $\log_7 1 = x$

      iv. $\log_8 x = 3$

      v. $\log_5(3x+1) = 2$

      vi. $\log_6(4x-7) = 0$

8. In this item we will examine the definition of the logarithmic function more closely. We have established that $y = b^x$ and $y = \log_b x$ are inverse functions. Remember that only one-to-one functions have inverses that are also functions and that the domains and ranges of inverse functions are interchanged.
   a. In defining the exponential function $y = b^x$, we stated that $b > 0$ and $b \neq 1$. Explain why these two facts must be true for $y = b^x$ to be an exponential function.

   b. If the base of the exponential function $y = b^x$ must be positive, what must be true about y (the range) of an exponential function. Explain.

   c. Use your responses to *parts a* and *b* to state any restrictions on the inverse of the exponential function, the function $y = \log_b x$.

CCGPS Mathematics III
Task 2: *Exponential Functions and Their Inverses*
Day 2 Homework

Express each logarithmic equation as an exponential equation and solve.

1. $\log_2\left(\frac{1}{32}\right) = x$

2. $\log_3\left(x^2 + x + 3\right) = 2$

3. $\log_x 5 = \frac{1}{3}$

4. $\log_x 8 = \frac{-3}{4}$

5. The time t, in seconds, at which the current in a circuit is I amperes can be calculated using the formula $t = -\log_2 I$. After how many seconds will the current be .25 amperes?

CCGPS Mathematics III
Task 2: ***Exponential Functions and Their Inverses*** **Day 3/4**
GaDOE *The Population of Exponentia # 3f, and #4b*

GPS Standard(s):
MM3A3. Students will solve a variety of equations and inequalities.
b. Solve…exponential and logarithmic equations analytically, graphically, and using appropriate technology.
d. Solve a variety of types of equations by appropriate means choosing among mental calculation, pencil and paper, or appropriate technology.

New vocabulary: common logarithms

Mathematical concepts/skills:
- writing exponential equations with a base of 10 as logarithmic equations with the common base
- using common logarithms to solve exponential equations
- solving exponential equations by using common logarithms to write both sides of the equation as 10 raised to a power

Prior knowledge:
- properties of exponents

Essential question(s): How can I solve exponential equations using common logarithms?

Suggested materials:
- graphing calculators
- graph paper

Warm-up: Post the following:

Write $10^x = 12$ as a logarithmic equation.

Opening: Discuss the warm-up. Conduct a mini-lesson on common logarithms and then have students use their calculators to *solve* the equation introduce in the *Warm-up*.

$$10^x = 12 \rightarrow x = \log_{10} 12 \rightarrow x \approx 1.079$$

Worktime: Students should work in pairs to complete *Items 9 - 11* of the task.

Allow students to complete *Item 9* and share responses to this item.

Explain the procedure for solving the equation $2^x = 100$ and then allow students to complete *Items 10* and *11*.

Closing: Allow students to share their responses to *Items 10* and *11*.

Homework: See attached.

Differentiated support/enrichment:

Check for understanding:

Resources/materials for Math Support:

CCGPS Mathematics III
Task 2: ***Exponential Functions and Their Inverses***
Day 3 Student Task

For any logarithmic expression $\log_b x$, as with exponential expressions, the base must be a positive number. Logarithms which use 10 for the base are called **common logarithms** and may be denoted simply as ***log x***. (It is not necessary to write the base). The *log* button on your calculator computes common logarithms.

Understanding common logarithms can help you solve more complex exponential equations.

Suppose you were asked to solve the equation $10^x = 250$.

We know that $10^2 = 100$ and $10^3 = 1000$, so x should be between 2 and 3. Rewriting the exponential equation as a logarithmic equation with a common logarithm can help solve the problem.

$$10^x = 250 \rightarrow \log 250 = x$$

Using the calculator, we find that $x \approx 2.3979$

9. Use common logarithms to solve each of the following equations for x.
   a. $10^x = 15$

   b. $10^x = 0.3458$

   c. $3(10^x) = 2345$

   d. $-2(10^x) = -6538$

To this point you have been able to solve exponential equations *algebraically* only if you could write both side of the equation in terms of the same base. For example, you could solve $2^x = 8$ but not $2^x = 9$. Numerous times in this unit, you have solved exponential equations numerically (using tables) and graphically. You will now be able to solve those equations algebraically as well.

Consider the following problem: A bacteria culture in a Petri dish begins with a single bacterium and doubles every hour. Once the population reaches 100, the bacteria begin dying. How long does it take the population to reach 100?

From our earlier work we know the function that models this growth can be written as $f(x) = 2^x$. The equation we need to solve will then be $2^x = 100$.

We know that $2^6 = 64$ and $2^7 = 128$, so the answer must be between 6 and 7.

If we rewrite both 2 and 100 as a power with a base of 10, we have $10^a = 2$ and $10^b = 100$.

$$10^a = 2 \rightarrow a = \log 2 \rightarrow a \approx .3010$$
$$10^b = 100 \rightarrow b = \log 100 \rightarrow b = 2$$

Now, replacing 2 with $10^{0.301}$ and 100 with 10^2, we have the following:

$(10^{0.301})^x = 10^2$	
$10^{0.301x} = 10^2$	*Power of a power property*
$0.301x = 2$	*Common base property*
$x \approx 6.6439$	

Explain why you might be confident that this is the correct answer.

10. Solve each of the following equations, taken from earlier explorations, using the method described above.
    a. $2 = 1.1^x$ (Steps *i – iv* will help you solve this first problem.)
        i. $10^a = 2$ and $10^b = 1.1$. Find *a* and *b*.
        ii. Use substitution to write an equation in terms of powers of 10.
        iii. Use the power-to-power and common base properties.
        iv. Solve for *x*.
    b. $300(0.8)^x = 10$
    c. $100 = 60(2^x)$

11. Use the method described above to find the zeros of $f(x) = 2^x - 5$. Verify your work using your graphing calculator.

CCGPS Mathematics III
Task 2: *Exponential Functions and Their Inverses*
Day 3 Homework

Evaluate each of the following *without* using your calculator.

1. log 10000
2. log 0.01
3. log 10
4. log 1
5. log 10^7

Evaluate each of the following using your calculator.

6. log 0.004
7. log ½

Use common logarithms to solve each equation.

8. $10^x = 124$
9. $-2(10^{2x}) = 1560$

Solve the following equations by writing both sides as a power of 10.

10. $3^x = 500$
11. $125(2^x) - 12 = 63$

12. The amount of drug in a patient's bloodstream peaks at 500 mg and then decreases by 30% every 5 hours.
    a. Write a function for the amount of blood remaining in the bloodstream *t* hours after the peak level.
    b. How long after the peak will it take for the drug level to drop to 200 mg?

CCGPS Mathematics III
Task 2: *Exponential Functions and Their Inverses* **Day 4/4**
GaDOE *The Population of Exponentia # 5 - 7*

CCSS Standard(s):
Functions
Interpreting Functions F-IF
Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*

  e. Graph exponential and logarithmic functions, showing intercepts and end behavior.

Building Functions F-BF
Build new functions from existing functions

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

4. Find inverse functions.

  c. (+) Read values of an inverse function from a graph or a table, given that the function has an inverse.

5. (+) Understand the inverse relationship between exponents and logarithms and use this relationship to solve problems involving logarithms and exponents.

GPS Standard(s):
MM3A2. Students will explore logarithmic functions as inverses of exponential functions.

e. Investigate and explain characteristics of exponential and logarithmic functions including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, and rate of change.

f. Graph functions as transformations of $f(x) = a^x$, $f(x) = \log_a x$, $f(x) = e^x$, $f(x) = \ln x$.

New vocabulary:

Mathematical concepts/skills:

- investigating characteristics of exponential and logarithmic functions including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, and rates of change
- graphing functions as transformations of $f(x) = a^x$ and $f(x) = \log_a x$

Prior knowledge:
- investigating characteristics of functions including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, and rates of change
- graphing transformations of functions

Essential question(s): What are the characteristics of logarithmic functions and how are these characteristics affected by transformations of the functions?

Suggested materials:
- graphing calculators
- graph paper

Warm-up: Post the following:

Working with your partner, compare responses to homework. Be prepared to share your solutions and to ask any remaining questions you have related to the assignment.

Opening: Discuss solutions to the problems assigned as homework for the previous lesson. It is important that students understand how and when to use their calculators to evaluate common logarithms, how to convert between exponential and logarithmic equations, and how to solve equations using common logarithms.

The function representing the level of drug in the blood steam after *t* hours (problem 12) is $f(t) = 500(1-0.3)^{\frac{t}{5}}$ or $f(t) = 500(0.7)^{\frac{t}{5}}$. To determine how long after peak it takes for the level of drug to drop to 200 mg we write the equation $200 = 500(0.7)^{\frac{t}{5}}$.

$200 = 500(0.7)^{\frac{t}{5}} \rightarrow 0.4 = (0.7)^{\frac{t}{5}}$

Writing each side of the equation as a power of 10, we get

$10^a = .4 \rightarrow a = \log 0.4 \approx -0.3979$

$10^b = .7 \rightarrow b = \log 0.7 \approx -0.1549$

$10^{-0.3979} = \left(10^{-0.1549}\right)^{\frac{t}{5}} \rightarrow -0.3979 = \dfrac{-0.1549t}{5} \rightarrow t \approx 12.84 \text{ hours}$

After a discussion of the homework, ask students how they might graph $y = \log_5 x$ by hand. Give students time to attempt the graph and then graph the function as a class before beginning the worktime. It is important for students to see that $y = \log_5 x$ can be graphed by writing the equivalent exponential equation $5^y = x$ or by thinking of the inverse function $y = 5^x$ and interchanging *x* and *y*.

Worktime: Students should work in pairs to complete *Items 12 and 13* of the task.
.

Closing: Allow students to share responses to all parts of *Items 12* and *13*.

Homework:

Differentiated support/enrichment:

Check for understanding:

Resources/materials for Math Support:

- investigating characteristics of functions including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, and rates of change
- graphing transformations of functions

CCGPS Mathematics III
Task 2: *Exponential Functions and Their Inverses*
Day 4 Student Task

12. We want to make explicit the characteristics that are common to ALL general logarithmic functions, regardless of the base. To do this, we need to look at graphs of different logarithmic functions.

a. Graph $y = \log_5 x$ by hand. Make a table of values to help you and don't forget to use some x-values between 0 and 1. Why might you consider $y = 5^x$ in making your table?

b. Draw the graph of $y = \log_3 x$ by hand.

c. Compare the graphs of $y = \log_5 x$ and $y = \log_3 x$. Which one increases faster? How can you tell?

d. Use your graphing calculator and the graphs you drew by hand in *parts a* and *b* of this item to help you complete the following table.

	$y = \log_5 x$	$y = \log_3 x$	$y = \log_{\left(\frac{1}{3}\right)} x$	$y = \log_{\left(\frac{1}{5}\right)} x$
Domain				
Range				
Intercept				
Asymptote				
Zeros				
Increasing / Decreasing				

e. Use the information in your table to answer the following:

i. When do logarithmic functions increase? When do they decrease?

ii. Give an example of a function that increases faster than any of the functions in the table.

iii. Which of the above graphs decreased fastest?

iv. Give an example of a function that would decrease faster than any of the ones above. Explain why you chose this function.

13. In this item we will investigate transformations of $y = \log_b x$.

a. Consider $y = \log(x - 3)$.

i. What is the parent function for this transformation?

ii. Describe the transformation of the parent function in words.

iii. Sketch the graphs of both functions.

iv. For the parent function, give:
Domain ______________
Asymptote ____________
Intercepts _____________

v. For the transformation $y = \log(x - 3)$, give:
Domain ______________
Asymptote ____________
Intercepts _____________

b. Consider the function $y = \log_4(3x - 5)$.

i. To find the domain, we solve the inequality $3x - 5 > 0$. Say why.
ii. The asymptote can be determined by solving the equation $3x - 5 = 0$. Explain.

iii. Find the domain. ________ Find the asymptote. __________

iv. In general, how do we find x-intercepts of a function? Find the x-intercept for this function.

c. Find the domain, asymptote, x-intercept, and y-intercept (if applicable) of $f(x) = \log_5(-3x + 8)$.

Domain: ____________ Asymptote: ______________

x-intercept: __________ y-intercept: ______________

Task 3: Modeling with Exponential and Logarithmic Functions

CCGPS Mathematics III
Task 3: ***Modeling with Exponential and Logarithmic Functions*** **Day 1/3**
GaDOE *Modeling Natural Phenomena #1 -3*

CCSS Standard(s):
Number and Quantity
Quantities N-Q
Reason quantitatively and use units to solve problems.

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

2. Define appropriate quantities for the purpose of descriptive modeling.

The Real Number System N-RN
Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define* $5^{1/3}$ *to be the cube root of 5 because we want* $(5^{1/3})^3 = 5^{(1/3)3}$ *to hold, so* $(5^{1/3})^3$ *must equal 5.*

2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Algebra
Seeing Structure in Expressions A-SSE
Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.*

   a. Interpret parts of an expression, such as terms, factors, and coefficients.

   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of P and a factor not depending on P.*

Write expressions in equivalent forms to solve problems

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.*

   c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

Creating Equations* A-CED

Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable and use them to solve problems.

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

Functions

Interpreting Functions F-IF

Interpret functions that arise in applications in terms of the context

4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.**

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.**

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*

   e. Graph exponential and logarithmic functions, showing intercepts and end behavior.

Building Functions F-BF

Build a function that models a relationship between two quantities

1. Write a function that describes a relationship between two quantities.*

   a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

Linear, Quadratic, and Exponential Models* F –LE

Construct and compare linear, quadratic, and exponential models and solve problems

4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Interpret expressions for functions in terms of the situation they model

5. Interpret the parameters in a linear or exponential function in terms of a context.

GPS Standard(s):
MM3A2. Students will explore logarithmic functions as inverses of exponential functions.
g. Explore real phenomena related to exponential and logarithmic functions including half-life and doubling time.

MM3A3. Students will solve a variety of equations and inequalities.
b. Solve exponential and logarithmic equations analytically, graphically, and using appropriate technology.
New vocabulary: pH scale, decibel, Richter scale

Mathematical concepts/skills:
- using logarithmic functions to model natural phenomena including pH levels, magnitude of sound, and the magnitudes of earthquakes
- solving exponential and logarithmic equations graphically and algebraically
- explaining characteristics of logarithmic functions including domain, range, asymptotes, zeros, intercepts, intervals of increase and decrease, and rates of change
- graphing functions as transformations of $f(x) = \log_a x$

Prior knowledge:
- solving exponential equations graphically and algebraically
- explaining characteristics of functions including domain, range, asymptotes, zeros, intercepts, intervals of increase and decrease, and rates of change
- graphing functions as transformations of $f(x)$

Essential question(s): How can I use exponential and logarithmic functions to model and solve real-world problems?

Suggested materials:
- graphing calculators
- graph paper

Warm-up: Post the following:

Read silently the explanation of the pH scale given at the beginning of the task. Make notes of any information you feel may be important in answering questions related to this phenomena.

Opening: Discuss the pH scale making sure that all students understand the pertinent information.

Worktime: Students should work in pairs to complete *Items 1 - 3* of the task. *Item 4* may be addressed in class, if time permits, or assigned as homework.

As they complete each item, allow students to share their responses before moving to a new scenario.

Closing: Allow students to share their responses to any remaining items. (See GaDOE teacher Notes.)

Homework:

Differentiated support/enrichment:

Check for understanding:

Resources/materials for Math Support: Students should preview concepts and formulas related to pH scale, magnitude of sound, and the Richter scale.

CCGPS Mathematics III
Task 3: *Modeling with Exponential and Logarithmic Functions*
Day 1 Student Task

Logarithmic and exponential functions are used to model a wide variety of problems in many different fields. In this task we will investigate some of those problems.

Modeling Natural Phenomena
pH is a measure of the acidity or alkalinity of a solution. The pH scale takes its name from the words *potential of hydrogen* and measures the number of hydrogen ions, $[H^+]$ in a substance. The scale generally ranges from 0 to 14. A solution that has a pH less than 7 is described as acidic and a solution that has a pH greater than 7 is described as alkaline or basic. A solution with a pH of 7 is neutral.

Why is pH significant? pH influences the structure and the function of many enzymes (protein catalysts) in living systems. Overall, most human cell enzymes work best in a slightly alkaline environment of about 7.4. Cellular pH is so important that death may ensue within hours if a person becomes acidotic. One such example is *unregulated diabetes* — high blood sugar occurs and acids can form that rapidly destroy enzymes and cells.

$[H^+]$ is usually a very large or very small number. Because logarithms can be used to represent very large or very small numbers, we use logarithms to determine pH. The pH is the negative logarithm of the concentration of free hydrogen ions, measured in moles per liter (moles/L).

1. The conversion formula is $pH = -\log [H^+]$.[1]

    a. Consider the general common logarithmic function, $f(x) = \log x$. How will the graph of the pH conversion function, $g(x) = -\log x$, differ from the graph of $f(x)$? Compare domains, ranges, intercepts, asymptotes, and intervals of increase and decrease. What kind of graphical transformation is this?

    b. If a water sample has a pH of 5, what is the concentration of hydrogen ions in the sample?

    c. Suppose a water sample has a pH of 7. How does the concentration of hydrogen ions in this sample compare to the concentration of hydrogen ions in the sample in *part b*? Explain.

    d. The normal $[H^+]$ range for drinking water is between approximately 10^{-6} and 3.16×10^{-9}. What is the approximate pH range?

    e. For a certain substance, the concentration of hydrogen ions measured 10^{-4}. Is this more or less acidic than normal drinking water? Explain.

The magnitude of sound D, in decibels, is determined using the formula $D = 10\log\frac{I}{I_0}$, where I is the intensity of sound and I_0 is the intensity of the threshold of human hearing. The intensity of the threshold of human hearing is 10^{-16} watts per square centimeter (W/cm^2). This is the intensity of the faintest sound the human ear can detect. The intensity of the loudest sound the ear can detect (causing great pain by the way) is 10^{-4}.

2. Consider the information above related to the magnitude of sound.
   a. What is the relationship between the intensity of the *threshold* of human hearing, 10^{-16} W/cm^2, and the intensity of the loudest sound the ear can detect, 10^{-4} W/cm^2?

   b. What is the loudness or magnitude of each sound to the nearest decibel?
      i. the threshold of hearing, $I = 10^{-16}$.
      ii. a whisper, $I = 3.16 \text{ x } 10^{-15}$
      iii. a subway train, $I = 5.01 \text{ x } 10^{-7}$
      iv. the average rock band, $I = 10^{-5}$

   c. Instant perforation of the eardrum occurs at a decibel level of 1016. Find the intensity of sound (I) for this decibel level (D).

The Richter magnitude scale assigns a single number to quantify the amount of seismic energy released by an earthquake. For each increase of one point on the Richter scale, the relative size or *shaking amplitude* of an earthquake increases by a factor of 10.

3. Complete the following table that shows the correspondence of relative size and Richter scale number.

Richter Scale Number	Relative Size
1	10
2	
3	
…	…
6	

   a. Write a logarithmic equation for the relationship between Richter number and relative size of an earthquake. (Let relative size, *s,* be your independent variable and Richter scale number, *r*, be the dependent variable.)What is the name for this special type of logarithm?

b. Using the equation you wrote in *part a*, find the relative size of an earthquake that measures 8 on the Richter scale. Find the relative size of an earthquake that measures 2. Compare the magnitudes of the two earthquakes.

c. In 2002, an earthquake of magnitude 7.9, one of the largest in the U.S., occurred in Denali National Park in Alaska. What was the relative size of this earthquake?

d. On April 29, 2003, an earthquake in Fort Payne, Alabama was felt by many residents of northern Georgia. The magnitude was 4.6. How does the relative size of the Alabama earthquake compare with the relative size of the Denali earthquake?

4. Rather than discuss relative size, we often prefer to discuss the amount of seismic energy released by an earthquake. A formula that relates the number on a Richter scale to the energy of an earthquake is $r = 0.67 \log E - 7.6$, where r is the number on the Richter scale and E is the energy in ergs.

   a. Describe how the formula for r transforms the graph of $f(x) = \log x$.

   b. Determine the domain, range, intercepts, and asymptotes for the graph of $r = 0.67 \log E - 7.6$. Sketch the graph.

   domain: _________________ range: _________________

   asymptote: ______________ intercept(s): _____________

   c. What is the Richter number of an earthquake that releases 3.9×10^{15} ergs of energy? (Be careful when inputting this into the calculator.)

   d. How much energy was released by the 2002 Denali earthquake? By the 2003 Alabama earthquake?

CCGPS Mathematics III
Task 3: *Modeling with Exponential and Logarithmic Functions* **Day 2/3**
GaDOE *Modeling Natural Phenomena #4b and d, #5c-f*

CCSS Standard(s):
Number and Quantity
Quantities N-Q
Reason quantitatively and use units to solve problems.

1. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

2. Define appropriate quantities for the purpose of descriptive modeling.

The Real Number System N-RN
Extend the properties of exponents to rational exponents.

1. Explain how the definition of the meaning of rational exponents follows from extending the properties of integer exponents to those values, allowing for a notation for radicals in terms of rational exponents. *For example, we define* $5^{1/3}$ *to be the cube root of 5 because we want* $(5^{1/3})^3 = 5^{(1/3)3}$ *to hold, so* $(5^{1/3})^3$ *must equal 5.*

Algebra
Seeing Structure in Expressions A-SSE
Interpret the structure of expressions

1. Interpret expressions that represent a quantity in terms of its context.*

   a. Interpret parts of an expression, such as terms, factors, and coefficients.

   b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of P and a factor not depending on P.*

Write expressions in equivalent forms to solve problems

3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.*

   c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

Creating Equations* A-CED
Create equations that describe numbers or relationships

1. Create equations and inequalities in one variable and use them to solve problems.

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

Functions

Building Functions F-BF

Build a function that models a relationship between two quantities

1. Write a function that describes a relationship between two quantities.*

   a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

Linear, Quadratic, and Exponential Models* F –LE

Construct and compare linear, quadratic, and exponential models and solve problems

4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Interpret expressions for functions in terms of the situation they model

5. Interpret the parameters in a linear or exponential function in terms of a context.

GPS Standard(s):

MM3A2. Students will explore logarithmic functions as inverses of exponential functions.

g. Explore real phenomena related to exponential and logarithmic functions including half-life and doubling time.

MM3A3. Students will solve a variety of equations and inequalities.

b. Solve exponential and logarithmic equations analytically, graphically, and using appropriate technology.

New vocabulary: Carbon-14 dating, the transcendental number e

Mathematical concepts/skills:

- using logarithmic and exponential functions to model real world situations including Carbon-14 dating and compound interest
- solving exponential and logarithmic equations algebraically and using appropriate technology

Prior knowledge:

- solving simple exponential equations

Essential question(s): How can I use exponential and logarithmic functions to model and solve real-world problems?

Suggested materials:

- graphing calculators
- graph paper

Warm-up: Post the following:

> *Read silently the explanation of Carbon-14 dating given at the beginning of the task. Make notes of any information you feel may be important in answering questions related to this phenomena.*

Opening: Discuss Carbon dating making sure that all students understand the pertinent information.

Worktime: Students should work in pairs to complete *Items 5 and 6* of the task.

When students complete *Item 5*, allow them to share their responses before moving on to *Item 6*.

Closing: Allow students to share their responses to Item 6. (See GaDOE teacher Notes.)

Homework:

Differentiated support/enrichment:

Check for understanding:

Resources/materials for Math Support: Students should preview concepts and formulas related to Carbon-14 dating and compound interest.

CCGPS Mathematics III
Task 3: ***Modeling with Exponential and Logarithmic Functions***
Day 2 Student Task

Scientists use carbon-dating to determine the ages of carbon-based substances. The isotope Carbon-14 (C14) is widely used in radiocarbon dating. This form of carbon is formed when plants absorb atmospheric carbon dioxide into their organic material during photosynthesis. After plants die, no more C14 is formed and the C14 in the material declines exponentially.

1. The half-life of the C14 isotope, the amount of time it takes for half of the C14 to decay, is approximately 5730 ± 40 years. This is known as the **Cambridge half-life.**
    a. Find an equation that models the part of the initial C14 remaining in a carbon-based substance t years after the death of the subject. Use 5730 as the half-life of Carbon-14.
    b. How much of the original C14 remains in a fossil that is 4000 years old?
    c. A plant contains 64.47% of its original Carbon-14. Approximately how long ago did it die?

Business

In Math 2, you learned how to calculate investments compounded over different time periods using the function $A(t) = A_0\left(1+\frac{r}{n}\right)^{nt}$, where $A(t)$ is the final amount of the investment after t years, r is the annual rate of interest expressed as a decimal, n is the number of times the investment is compounded per year, and t is the time in years.

You also learned how to calculate the value of investments compounded continuously using the formula $A(t) = A_0e^{rt}$ where $A(t)$ is the final amount of the investment after t years, r is the annual rate of interest expressed as a decimal, and t is the time in years The formula for continuous compounding employs the **transcendental number** $e \approx 2.71828$.

2. Assume that you have $10,000 to invest.
    a. Complete the following chart to show how much you would earn if her money was invested in each of the specified accounts for 10 years.

Frequency of compounding	Annual interest rate	Formula with substituted values	Amount after 10 years
1. Quarterly	3.65%		
2. Monthly	3.65%		
3. Daily	3.6%		
4. Continuously	3.6%		

Which account would you choose?

b. Suppose you are particularly interested in how long it will take you to double your money.

Consider the first option. If your money doubles, you will have $20,000. So the equation you want to solve is

$$20,000 = 10,000\left(1+\frac{0.0365}{4}\right)^{(4t)} \rightarrow 2 = 1\left(1+\frac{0.0365}{4}\right)^{(4t)} \rightarrow 2 = (1.009125)^{4t}.$$

Remember that to solve some exponential equations, you must employ the inverse operation, logarithms. Recall that we can write each base as 10 to a power.

$2 = 10^a$ → log 2 = a → a = ___________

1.009125 = 10^b → log 1.009125 = b → b = _____________

2 = $(1.009125)^{4t}$ → $10^{\_\_\_} = 10^{\_\_\_\,4t}$ → ______ =_____t → t = _______ years

c. Determine how long it will take you to double your money under options 2 and 3. Show your work.

CCGPS Mathematics III
Task 3: *Modeling with Exponential and Logarithmic Functions* **Day 3/3**
GaDOE *Modeling Natural Phenomena #5e - g*

CCSS Standard(s):
The Real Number System N-RN
Extend the properties of exponents to rational exponents.
2. Rewrite expressions involving radicals and rational exponents using the properties of exponents.

Algebra
Seeing Structure in Expressions A-SSE
Interpret the structure of expressions
1. Interpret expressions that represent a quantity in terms of its context.*

a. Interpret parts of an expression, such as terms, factors, and coefficients.

b. Interpret complicated expressions by viewing one or more of their parts as a single entity. *For example, interpret* $P(1+r)^n$ *as the product of P and a factor not depending on P.*

Write expressions in equivalent forms to solve problems
3. Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.*

c. Use the properties of exponents to transform expressions for exponential functions. *For example the expression* 1.15^t *can be rewritten as* $(1.15^{1/12})^{12t} \approx 1.012^{12t}$ *to reveal the approximate equivalent monthly interest rate if the annual rate is 15%.*

Creating Equations* A-CED
Create equations that describe numbers or relationships
1. Create equations and inequalities in one variable and use them to solve problems.

2. Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

Functions
Interpreting Functions F-IF
Interpret functions that arise in applications in terms of the context
4. For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. *Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.**

5. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines in a factory, then the positive integers would be an appropriate domain for the function.**

Analyze functions using different representations

7. Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*

   e. Graph exponential and logarithmic functions, showing intercepts and end behavior.

Building Functions F-BF

Build a function that models a relationship between two quantities

1. Write a function that describes a relationship between two quantities.*

   a. Determine an explicit expression, a recursive process, or steps for calculation from a context.

Build new functions from existing functions

3. Identify the effect on the graph of replacing $f(x)$ by $f(x) + k$, $k\,f(x)$, $f(kx)$, and $f(x + k)$ for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. *Include recognizing even and odd functions from their graphs and algebraic expressions for them.*

Linear, Quadratic, and Exponential Models* F –LE

Construct and compare linear, quadratic, and exponential models and solve problems

4. For exponential models, express as a logarithm the solution to $ab^{ct} = d$ where a, c, and d are numbers and the base b is 2, 10, or e; evaluate the logarithm using technology.

Interpret expressions for functions in terms of the situation they model

5. Interpret the parameters in a linear or exponential function in terms of a context.

GPS Standard(s):

MM3A2. Students will explore logarithmic functions as inverses of exponential functions.

c. Define logarithmic functions as inverses of exponential functions.

e. Investigate and explain characteristics of exponential and logarithmic functions including domain and range, asymptotes, zeros, intercepts, intervals of increase and decrease, and rate of change.

g. Explore real phenomena related to exponential and logarithmic functions including half-life and doubling time.

MM3A3. Students will solve a variety of equations and inequalities.

b. Solve exponential and logarithmic equations analytically, graphically, and using appropriate technology.

New vocabulary: natural logarithm, $\ln x$

Mathematical concepts/skills:

- using logarithmic and exponential functions to model natural phenomena
- using the natural logarithm to solve exponential equations with base e
- solving exponential and logarithmic equations graphically and algebraically
- explaining characteristics of the graph of $f(x) = e^x$ including domain, range, asymptotes, zeros, intercepts, intervals of increase and decrease, and rates of change
- graphing functions as transformations of $f(x) = a^x$

Prior knowledge:

- solving exponential equations graphically and algebraically
- explaining characteristics of functions including domain, range, asymptotes, zeros, intercepts, intervals of increase and decrease, and rates of change
- graphing functions as transformations of $f(x)$

Essential question(s): How can I use exponential and logarithmic functions to model and solve real-world problems?

Suggested materials:

- graphing calculators
- graph paper

Warm-up: Post the following:

Evaluate the following expressions without using you calculator. Show your work.

1. $\log_e e$
2. $\log_e e^x$

Opening: Allow students to discuss responses to the warm-up. Conduct a short mini-lesson on the natural logarithm and its notation.

Worktime: Students should work in pairs to complete *Items 7 and 8* of the task.

After students have had ample time to complete *Item 7*, allow them to share responses.

Closing: Allow students to share their responses to *Item 8*. An Anchor Chart of the graph of $f(x) = e^x$ and its inverse $f(x) = \ln x$ may be beneficial. (See GaDOE teacher Notes.)

Homework:

Differentiated support/enrichment:

Check for understanding:

Resources/materials for Math Support:

CCGPS Mathematics III
Task 3: ***Modeling with Exponential and Logarithmic Functions***
Day 3 Student Task

7. You also want to determine how long it would take to double your money if the money is compounded continuously. The equation you want to solve is:

$$20{,}000 = 10{,}000e^{0.036t} \rightarrow 2 = e^{0.036t}.$$

Although you can employ the same method of writing each base as 10 to a power, taking advantage of common logarithms, generally when we have a base of *e*, we use **natural logarithms** instead. A natural logarithm is a logarithm with a base of *e*.

The natural logarithmic function is $f(x) = \log_e x \; or \; y = \log_e x$. Remembering the definition of logarithms, we know that $y = \log_e x \Leftrightarrow e^y = x.$ Like the common logarithm, there is a special notation for the natural logarithmic function, $\log_e x = \ln x$.

a. Evaluate the following expressions without using you calculator:

i. $\log_e e$

ii. $\ln_e e$

iii. $\ln e^3$

iv. $\ln e^{-5}$

b. Use properties of inverse functions to explain your results in *part a*.

c. So back to your problem of determining how long it would take to double your money if the money is compounded continuously. To solve the equation

$2 = e^{0.036t}$ we take the natural log of both sides.

$$2 = e^{0.036t} \rightarrow \ln 2 = \ln e^{0.036t} \rightarrow \ln 2 = 0.036t$$

Use your calculator to help you finish solving the equation for *t*.

d. Suppose you invested your savings in an account that was compounded continuously at an annual interest rate of 3.7%. How long would it take your money to double?

8. The function $f(x) = e^x$ is a special case of the function $f(x) = a^x$.

a. Sketch a graph of $f(x) = e^x$ and list the domain, range, intercepts, and asymptotes.

b. Suppose you invest \$1 in an account that is compounded continuously at an annual interest rate of 3%. Write a function that represents the amount of money you will have at the end of *t* years and sketch its graph.

c. Suppose you invest \$10,000 in the account described in *part b*. Write a function that represents the amount of money you will have at the end of *t* years. How will the graph of this function be different from the graph in *part b*? Do any of the characteristics of the original graph change? What kind of transformation is this?

d. Suppose that you receive a \$5,000 bonus and add this amount to your investment on the day you decide to remove your investment from the bank. Write a function that represents the amount of money you will have at the end of *t* years. How will the graph of this function be different from the graph in *part b*?